

# The Non-Metricity Formulation of General Relativity

Igor Mol\*

Institute of Mathematics, Statistics and Scientific Computation  
Unicamp, SP, Brazil

January 14, 2015

## Abstract

After recalling the differential geometry of non-metric connections in the formalism of differential forms, we introduce the idea of a Non-Metricity (NM) connection, whose connection 1-forms coincides with the non-metricity 1-forms for a class of cobase fields. Then we formulate a theory of gravitation (equivalent to General Relativity (GR)) which admits a geometrical interpretation in a flat torsionless space where the gravitational field is completely manifest in the non-metricity of a NM connection.

We define and then apply the non-metricity gauge to a gravitational Lagrangian density discovered by Wallner [11] (proved in Appendix A to be equivalent to Einstein-Hilbert). The Einstein equations coupled to the matter currents ( $\mathcal{J}_\alpha$ ) thus becomes  $\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha$ , where ( $\mathcal{T}_\alpha$ ) is identified as the gravitational energy-momentum currents, to which we shall find a relatively simple and physically appealing form. It is also shown that in the gravitational analogue of the Lorenz gauge, our field equations can be written as a system of Proca equations, which may be of interest in the study of propagation of gravitational-electromagnetic waves.

---

\*igormol@ime.unicamp.br or igormol@gmail.com.

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Non-Metric Connections</b>	<b>4</b>
2.1	Basic Definitions . . . . .	4
2.2	Useful Identities . . . . .	6
2.3	The Non-Metricity Connection $\mathfrak{D}$ . . . . .	10
<b>3</b>	<b>Gravitation and Non-Metricity</b>	<b>12</b>
3.1	Field Equations . . . . .	12
3.2	Gravitation as Non-Metricity . . . . .	23
<b>4</b>	<b>Discussion</b>	<b>27</b>
<b>A</b>	<b>Einstein-Hilbert Lagrangian</b>	<b>30</b>

## 1 Introduction

In this paper, a theory of gravitation equivalent to General Relativity (GR) will be formulated, for which a geometrical interpretation where the gravitational field is manifest in the non-metricity of a flat torsionless connection is naturally attributed.

In order to do so, it will be necessary to recall some facts about the differential geometry of non-metric connections in parallelizable manifolds, which are presented in sections 2.1 and 2.2 in the formalism of differential forms<sup>1</sup>. Also, we introduce the concept of Non-Metricity (NM) connections, having the property that its connection 1-forms coincides with its non-metricity 1-forms relatively to a class of cobase fields, as described in section 2.3.

Subsequently to the mathematical preliminaries of section 2, we formulate our gravitational theory in section 3. In section 3.1, we start from a gravitational Lagrangian density  $\mathcal{L}$  discovered by Wallner [11], which is given in terms of a cobase field  $(g_\alpha) \in \bigwedge^1 M$  representing the gravitational potentials by

$$\mathcal{L} = \frac{1}{2} g_\alpha \wedge dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) - \frac{1}{4} g_\alpha \wedge dg^\alpha \wedge \star (g_\beta \wedge dg^\beta).$$

In Appendix A, the equivalence between the Wallner Lagrangian (WL) density and the Einstein-Hilbert Lagrangian density is established.

Then, also in section 3.1, we introduce the *non-metricity gauge*, whose geometrical meaning will become clear from the discussion of NM connections

---

<sup>1</sup>In section 2.2, we also derive an identity decomposing the connection 1-forms of an arbitrary connection in terms of its non-metricity 1-forms, its torsion 2-forms and some Levi-Civita connection 1-forms, something which may be useful to the study of gravitational theories with additional degrees of freedom [23].

presented in section 2.3. It will be shown from the variational principle (Proposition 33) that the gravitational field equations assumes the form

$$\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha,$$

where  $(\mathcal{J}_\alpha) \in \bigwedge^1 M$  are the matter energy-momentum currents and  $(\mathcal{T}_\alpha) \in \bigwedge^1 M$  are identified with the gravitational energy-momentum currents. In the non-metricity gauge, we find a relatively<sup>2</sup> simple expression for the gravitational energy-momentum currents, namely,

$$\begin{aligned} \mathcal{T}_\alpha &= \frac{1}{2} \star (dg_\beta \wedge i_\alpha \star dg^\beta - i_\alpha dg_\beta \wedge \star dg^\beta) \\ &\quad + \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \wedge g_\alpha + i_\beta \delta g^\beta \wedge g_\alpha - i_\alpha \delta g^\beta \wedge g_\beta, \end{aligned}$$

whose physical significance is discussed in the many Remarks and Examples of section 3.1.

In particular, we prove that if the gravitational Lorenz gauge  $\delta g_\alpha = 0$  ( $0 \leq \alpha \leq 3$ ) is adopted, the gravitational equations becomes the following system of coupled Proca equations with variable mass,

$$\square g_\alpha + \frac{1}{2} \langle dg_\beta | dg^\beta \rangle g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) + \mathcal{J}_\alpha.$$

This latter form of the field equations may be of interest in the study of the propagation of gravitational-electromagnetic waves, as briefly outlined in Examples 37 and 38.

As another straightforward application of the field equations, we also derive a force law for the matter currents coupled to the gravitational field. By identifying the 1-form

$$\mathcal{W}_\xi = \frac{1}{2} \star (dg_\beta \wedge i_\xi \star dg^\beta - i_\xi dg_\beta \wedge \star dg^\beta)$$

with the gravitational energy-flow along a Killing vector field  $\xi \in \sec TM$ , we easily prove that

$$\delta \mathcal{W}_\xi = \langle i_\xi dg_\alpha | \mathcal{T}^\alpha + \mathcal{J}^\alpha \rangle,$$

whose analogy with the Lorentz force law of electrodynamics is outlined in Remark 42.

In section 3.2, we finally show that the gravitational field equations can be completely rewritten in terms of the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms (defined in section 2.1) of a NM connection, together with the matter energy-momentum currents, becoming

$$\begin{aligned} i_\nu \mathcal{J}_\mu &= \mathbf{Q}_{\mu[\alpha\nu];}{}^\alpha + \mathbf{Q}_{\alpha\nu}{}^\alpha{}_{;\mu} - \left[ \mathbf{Q}_{\alpha\beta}{}^{\alpha;\beta} + \frac{1}{2} \left( \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]} + \mathbf{Q}_{\alpha\beta}{}^\alpha \mathbf{Q}_\gamma{}^{\beta\gamma} \right) \right] \eta_{\mu\nu} \\ &\quad - \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu} - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_\nu{}^{[\alpha\beta]} - \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_\beta{}^{\alpha\beta}. \end{aligned}$$

The gravitational field may therefore be interpreted as the manifestation of the non-metricity of a flat and torsionless connection living in the spacetime manifold.

---

<sup>2</sup>If compared, for instance, with [17].

## 2 Non-Metric Connections

In section 2.1, we state the basic definitions of differential geometry in the language of differential forms and, in particular, we discuss the non-metricity 1-forms. Then, in section 2.2, many identities useful in the study of the gravitational field equations and non-metricity are derived. Lastly, in section 2.3, we introduce the Non-Metricity (NM) connections in a parallelizable manifold, as it will be employed in section 3.1 to define the non-metricity gauge.

### 2.1 Basic Definitions

On what follows, let  $M$  be a parallelizable manifold.

**Notation 1** Here and thereafter,  $(f_{\alpha\beta\dots}) \in F$  signify a sequence  $(f_{\alpha\beta\dots})_{\alpha\beta\dots}$  of elements of the family  $F$ . Greek indexes always belongs to  $\{0, 1, \dots, \dim(M)-1\}$ .

**Definition 2** Recall that  $(E_\alpha) \in \sec TM$  is called a base field if every  $X \in \sec TM$  can be written as a linear combination of  $(E_\alpha)$ . On the other hand,  $(E_\alpha)$  is called a frame field if  $(E_\alpha)$  is a base field orthonormal according to a given metric [1] [2]. Also,  $(g_\alpha) \in \sec T^*M$  is called the cobase field of  $(E_\alpha)$  if  $g_\alpha(E_\beta) = \delta_{\alpha\beta}$  for all  $0 \leq \alpha, \beta \leq \dim(M)-1$ , and  $(g_\alpha)$  is a coframe field if  $(E_\alpha)$  is a frame field. For brevity, a frame field and a cobase field will be referred to just as a frame and a cobase, respectively.

**Notation 3**  $\bigwedge^p M \equiv \sec \bigwedge^p T^*M$  is the space of the sections of the bundle differential  $p$ -forms.

**Notation 4** Let  $\mathbf{g}$  be a metric tensor. We shall write  $g(X, Y) = \langle X|Y \rangle$  for all  $X, Y \in \sec TM$ .

Let  $(E_\alpha)$  a base field on  $M$  and  $(g_\alpha)$  its cobase field. The metric induced by  $(g_\alpha)$  is  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$ , and the pair  $(M, \mathbf{g})$  is called a pseudo-Riemmanian manifold. Here,

$$(\eta_{\alpha\beta}) = \text{diag}(-1, \dots, -1, +1, \dots, +1)$$

posses  $p$  negative and  $q$  positive eigenvalues. Recall that that  $\mathbf{g}$  is called Lorentzian when  $p = 1$  or  $q = 1$ , and we shall adopt the former convention.

**Notation 5** We denote by  $D$  an arbitrary connection on  $M$ , while  $\nabla$  is reserved to the Levi-Civita connection of  $(M, \mathbf{g})$ . The triple  $(M, \mathbf{g}, \nabla)$  is referred to as a pseudo-Riemannian space.

Let  $T$  be the torsion and  $R$  the Riemman curvature tensor of  $D$ , and  $(\omega^\alpha_\beta) \in \bigwedge^1 M$  the connection 1-forms of  $D$  relative to the cobase  $(g_\alpha)$ . Recall that for all  $X \in \sec TM$ ,

$$D_X E_\alpha = \omega^\beta_\alpha(X) E_\beta.$$

Since we shall use constantly the Levi-Civita connection to apply some tricks of differential forms, we must distinguish its connection 1-forms to that of an arbitrary connection. So we introduce the following notation.

**Notation 6** *The connection 1-forms of the Levi-Civita connection  $\nabla$  will be denoted by  $(\theta^\alpha_\beta)$ , and they will be called the Levi-Civita connection 1-forms.*

Also, let  $(\mathbf{T}^\alpha)$  and  $(\mathcal{R}^\alpha_\beta) \in \bigwedge^2 M$  be the torsion and curvature 2-forms relative to the cobase  $(g_\alpha)$ , and recall that for all  $X, Y \in \sec TM$ ,

$$T(X, Y) = \mathbf{T}^\alpha(X, Y) E_\alpha, \quad R(X, Y) E_\alpha = \mathcal{R}^\beta_\alpha(X, Y) E_\beta.$$

The Ricci tensor  $Ric \in \sec T_2^0 M$  of  $D$  is defined to be (up to an arbitrary sign) the contraction of  $R$  such that, for all  $X, Y \in \sec TM$ ,  $Ric(X, Y) = g^\alpha[R(X, E_\alpha) Y]$ . The Ricci 1-forms  $(\mathcal{R}_\alpha)$  are given by  $X \in \sec TM \mapsto \mathcal{R}_\alpha(X) = Ric(E_\alpha, X)$ . Therefore,

$$\begin{aligned} i_\beta \mathcal{R}_\alpha &= g^\gamma [R(E_\beta, E_\gamma) E_\alpha] = g^\gamma [\mathcal{R}^\delta_\alpha(E_\beta, E_\gamma) E_\delta] \\ &= \mathcal{R}^\gamma_\alpha(E_\beta, E_\gamma) = -i_\beta i_\gamma \mathcal{R}^\gamma_\alpha. \end{aligned}$$

Since this holds for all  $\beta$ , we have just proven that

$$\mathcal{R}_\alpha = -i_\beta \mathcal{R}^\beta_\alpha. \quad (1)$$

One can prove using the above definitions that the Cartan structural equations holds for an arbitrary connection  $D$  [1]. We shall, only for the sake of organization, state these equations as a Lemma.

**Lemma 7** *The connection 1-forms  $(\omega^\alpha_\beta)$  and the torsion and curvature 2-forms  $(\mathbf{T}^\alpha)$  and  $(\mathcal{R}^\alpha_\beta)$  of  $D$  relative to  $(g_\alpha)$  obeys the Cartan structural equations*

$$dg_\alpha = -\omega_a^\beta \wedge g_\beta + \mathbf{T}_\alpha, \quad (2)$$

$$\mathcal{R}^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\gamma \wedge \omega^\gamma_\beta. \quad (3)$$

Given the recurrent use of the first Cartan structural equation in the following developments, for the sake of brevity, it is worth to introduce the next notation.

**Notation 8**  $\mathcal{G}_\alpha = dg_\alpha - \mathbf{T}_\alpha \in \bigwedge^2 M$ ,  $0 \leq \alpha \leq \dim(M) - 1$ .

So now the first Cartan structural equation becomes just  $\mathcal{G}_\alpha = -\omega_a^\beta \wedge g_\beta$ .

The following definition of the non-metricity 1-forms  $(\mathcal{A}_{\alpha\beta})$  of  $D$  relative to the cobase  $(g_\alpha)$  is nonstandard (besides its inherent simplicity), so we shall be more careful in its formulation.

**Definition 9** *The non-metricity 1-forms  $(\mathcal{A}_{\alpha\beta}) \in \bigwedge^1 M$  of  $D$  relative to the cobase  $(g_\alpha)$  are such that*

$$\mathcal{A}_{\alpha\beta}(X) = -\frac{1}{2} (D_X g)(E_\alpha, E_\beta)$$

for all  $X \in \sec TM$ , while the non-metricity 2-forms  $(\mathbf{Q}_\gamma) \in \bigwedge^2 M$  are given by

$$\mathbf{Q}_\gamma = \frac{1}{2} \mathbf{Q}_{\alpha\beta\gamma} g^\alpha \wedge g^\beta,$$

where

$$\mathbf{Q}_{\alpha\beta\gamma} = i_{[\alpha} \mathcal{A}_{\beta]\gamma} \equiv i_\alpha \mathcal{A}_{\beta\gamma} - i_\beta \mathcal{A}_{\alpha\gamma}.$$

**Lemma 10** *The non-metricity 1-forms  $(\mathcal{A}_\beta^\alpha)$  are the symmetric part of the connection 1-forms  $(\omega_\beta^\alpha)$ , that is,*

$$\mathcal{A}_{\alpha\beta} = \frac{\omega_{\alpha\beta} + \omega_{\beta\alpha}}{2}.$$

In fact,  $0 = D_X [g(E_\alpha, E_\beta)] = (D_X g)(E_\alpha, E_\beta) + \langle D_X E_\alpha | E_\beta \rangle + \langle E_\alpha | D_X E_\beta \rangle = -2\mathcal{A}_{\alpha\beta}(X) + \langle \omega_\alpha^\gamma(X) E_\gamma | E_\beta \rangle + \langle E_\alpha | \omega_\beta^\gamma(X) E_\gamma \rangle = -2\mathcal{A}_{\alpha\beta}(X) + \omega_{\alpha\beta} + \omega_{\beta\alpha}$ , proving our Lemma. ■

We conclude this section by introducing a notation for the anti-symmetric part of  $(\omega_\beta^\alpha)$ , which will be related in an important identity to the connection 1-forms  $(\theta_\beta^\alpha)$  of the Levi-Civita connection.

**Definition 11** *Let  $(\omega_\beta^\alpha)$  be the connection 1-forms of some connection in  $M$ . The anti-symmetric part  $(\omega_{[\alpha\beta]})$  of  $(\omega_\beta^\alpha)$  are the 1-forms given by*

$$\omega_{[\alpha\beta]} = \frac{1}{2} (\omega_{\alpha\beta} - \omega_{\beta\alpha})$$

for all  $0 \leq \alpha, \beta \leq \dim(M) - 1$ .

Observe that, in the above notation,  $\omega_{\alpha\beta} = \omega_{[\alpha\beta]} + \mathcal{A}_{\alpha\beta}$ .

## 2.2 Useful Identities

Now we prove some identities which are useful to study the gravitational equations in the formalism of differential forms. We also derive the decomposition formula of the connection 1-forms of an arbitrary connection in  $M$  in terms of its non-metricity 1-forms, its torsion 2-forms and the Levi-Civita connection 1-forms.

The following Lemma gives an identity involving the codifferential of a cobase and the Levi-Civita connection 1-forms relative to that cobase. It will be used only as an intermediary step for the proof of the succeeding Lemmas. As the Levi-Civita connection always exists in the pseudo-Riemannian space  $(M, \mathbf{g})$ , its use will not imply in any loss of generality.

**Lemma 12** *The codifferential  $\delta g_\alpha$  and the Levi-Civita connection 1-forms  $(\theta_\beta^\alpha)$  relative to the cobase  $(g_\alpha)$  are related by*

$$\delta g_\alpha = i^\beta \theta_{\beta\alpha}.$$

To prove this, we shall need the following identity

$$d \star g_\alpha = -\theta_{\alpha\beta} \wedge \star g^\beta.$$

Indeed, let  $\varepsilon_{\alpha\beta\gamma\delta}$  be the Levi-Civita totally anti-symmetric symbol. So

$$\begin{aligned} d \star g_\alpha &= \frac{1}{3!} \varepsilon_{\alpha\beta\gamma\delta} d(g^\beta \wedge g^\gamma \wedge g^\delta) = dg^\beta \wedge \frac{1}{2} \varepsilon_{\alpha\beta\gamma\delta} g^\gamma \wedge g^\delta \\ &= dg^\beta \wedge \star(g_\alpha \wedge g_\beta) = -\theta^\beta_\gamma \wedge \star(g_\alpha \wedge g_\beta) \wedge g^\gamma. \end{aligned}$$

On the other hand,

$$\begin{aligned} \star(g_\alpha \wedge g_\beta) \wedge g^\gamma &= \star^2(\star(g_\alpha \wedge g_\beta) \wedge g^\gamma) = -\star i^\gamma(g_\alpha \wedge g_\beta) \\ &= -\star(\delta^\gamma_\alpha g_\beta - \delta^\gamma_\beta g_\alpha). \end{aligned}$$

Therefore  $d \star g_\alpha = \theta^\beta_\gamma \wedge \star(\delta^\gamma_\alpha g_\beta - \delta^\gamma_\beta g_\alpha) = \theta^\beta_\alpha \wedge \star g_\beta - \theta^\beta_\beta \wedge \star g_\alpha = -\theta_{\alpha\beta} \wedge \star g^\beta$ .

Now we can establish the Lemma. Just recall that

$$\begin{aligned} \delta g_\alpha &= -\star(d \star g_\alpha) = \star(\theta_{\alpha\beta} \wedge \star g^\beta) \\ &= \star(\star g^\beta \wedge \theta_{\beta\alpha}) = i_{\theta_{\beta\alpha}} \star^2 g^\beta = i^\beta \theta_{\beta\alpha}. \blacksquare \end{aligned}$$

The next Lemma utilizes the last one to give a connection-independent result, i.e., which holds in any parallelizable manifold<sup>3</sup>.

**Lemma 13** *The contraction of the differential of a cobase is related to the codifferential of that cobase by*

$$i^\alpha dg_\alpha = -\delta g^\alpha \wedge g_\alpha.$$

In particular, it follows that

$$i^\alpha dg_\alpha \wedge \star i^\beta dg_\beta = \delta g^\alpha \wedge \star \delta g_\alpha.$$

First,  $i^\alpha dg_\alpha = i^\alpha(-\theta_{\alpha\beta} \wedge g^\beta) = -i^\alpha \theta_{\alpha\beta} \wedge g^\beta + \theta_{\alpha\beta} g^{\alpha\beta} = -\delta g_\beta \wedge g^\beta$ .

Second, using that  $g^\alpha \wedge \star g^\beta = \star^2(\star g^\beta \wedge g^\alpha) = \star i^\alpha g^\beta = g^{\alpha\beta} \star 1$ ,

$$\begin{aligned} i^\alpha dg_\alpha \wedge \star i^\beta dg_\beta &= \delta g^\alpha \wedge g_\alpha \wedge \star(\delta g^\beta \wedge g_\beta) \\ &= \delta g^\alpha \wedge \delta g^\beta \wedge g_\alpha \wedge \star g_\beta \\ &= \delta g^\alpha \wedge \delta g^\beta \eta_{\alpha\beta} \star 1 \\ &= \delta g^\alpha \wedge \star \delta g_\alpha. \blacksquare \end{aligned}$$

The identity of the following Lemma is the bridge between the geometry of an arbitrary connection in  $M$  to the geometry of the Levi-Civita connection of  $(M, \mathbf{g})$ .

---

<sup>3</sup>Even if Lemma 12 employs a Levi-Civita connection, recall that in any parallelizable manifold we can construct a pseudo-Riemannian space by choosing a frame field and declaring it to be orthonormal.

**Lemma 14** *The anti-symmetric part  $\omega_{[\alpha\beta]}$  of the connection 1-forms  $(\omega^\alpha_\beta)$  of  $D$  relative to the cobase  $(g_\alpha)$  are given in terms of  $\mathcal{G}_\alpha = dg_\alpha - \mathbf{T}_\alpha$  (recall that  $(\mathbf{T}_\alpha)$  are the torsion 2-forms of  $D$ ) and of the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms  $\mathbf{Q}_\gamma$  of  $D$  by*

$$\omega_{[\alpha\beta]} = i_\beta \mathcal{G}_\alpha - i_\alpha \mathcal{G}_\beta + \frac{1}{2} i_\alpha i_\beta (g_\gamma \wedge \mathcal{G}^\gamma) - \mathbf{Q}_{\alpha\beta\gamma} \wedge g^\gamma.$$

Indeed, recall the first Cartan structural equation,  $\mathcal{G}_\gamma = dg_\gamma - \mathbf{T}_\gamma = -\omega_{\gamma\delta} \wedge g^\delta$ . Contraction gives  $i_\beta \mathcal{G}_\gamma = -i_\beta \omega_{\gamma\delta} \wedge g^\delta + \omega_{\gamma\beta}$ . Repeated contraction together with cyclic permutations yields

$$i_\alpha i_\beta \mathcal{G}_\gamma = i_\alpha \omega_{\gamma\beta} - i_\beta \omega_{\gamma\alpha}, \quad (4a)$$

$$i_\gamma i_\alpha \mathcal{G}_\beta = i_\gamma \omega_{\beta\alpha} - i_\alpha \omega_{\beta\gamma}, \quad (4b)$$

$$i_\beta i_\gamma \mathcal{G}_\alpha = i_\beta \omega_{\alpha\gamma} - i_\gamma \omega_{\alpha\beta}. \quad (4c)$$

After summing Eqs.(4b) and (4c), multiplying by  $-1$  and interchanging  $\beta$  with  $\gamma$  in  $i_\beta i_\gamma \mathcal{G}_\alpha$ , we obtain

$$i_\gamma (\omega_{\alpha\beta} - \omega_{\beta\alpha}) - i_\beta \omega_{\alpha\gamma} + i_\alpha \omega_{\beta\gamma} = i_\gamma i_\beta \mathcal{G}_\alpha - i_\gamma i_\alpha \mathcal{G}_\beta.$$

Now, summing with Eq.(4a), dividing by 2 and using the definitions of  $\omega_{[\alpha\beta]}$  (see Notation 11) and  $\mathcal{A}_{\alpha\beta}$  (see Lemma 10) yields

$$i_\gamma \omega_{[\alpha\beta]} + i_\alpha \mathcal{A}_{\beta\gamma} - i_\beta \mathcal{A}_{\alpha\gamma} = \frac{1}{2} (i_\gamma i_\beta \mathcal{G}_\alpha - i_\gamma i_\alpha \mathcal{G}_\beta + i_\alpha i_\beta \mathcal{G}_\gamma). \quad (5)$$

To remove the contraction  $i_\gamma$ , observe that

$$\begin{aligned} i_\alpha \mathcal{A}_{\beta\gamma} - i_\beta \mathcal{A}_{\alpha\gamma} &= \delta_\gamma^\delta (i_\alpha \mathcal{A}_{\beta\delta} - i_\beta \mathcal{A}_{\alpha\delta}) \\ &= i_\gamma (g^\delta \wedge (i_\alpha \mathcal{A}_{\beta\delta} - i_\beta \mathcal{A}_{\alpha\delta})) \\ &= i_\gamma (g^\delta \wedge \mathbf{Q}_{\alpha\beta\delta}), \end{aligned}$$

and that  $i_\alpha i_\beta \mathcal{G}_\gamma = \delta_\gamma^\delta i_\alpha i_\beta \mathcal{G}_\delta = i_\gamma (g^\delta \wedge i_\alpha i_\beta \mathcal{G}_\delta)$ . Therefore, Eq.(5) becomes

$$i_\gamma \omega_{[\alpha\beta]} = i_\gamma \left[ \frac{1}{2} (i_\beta \mathcal{G}_\alpha - i_\alpha \mathcal{G}_\beta + g^\delta \wedge i_\alpha i_\beta \mathcal{G}_\delta) - \mathbf{Q}_{\alpha\beta\delta} \wedge g^\delta \right].$$

Lastly, since this holds for any contraction  $i_\gamma$  and

$$\begin{aligned} g^\delta \wedge i_\alpha i_\beta \mathcal{G}_\delta &= -i_\alpha (g^\delta \wedge i_\beta \mathcal{G}_\delta) + i_\beta \mathcal{G}_\alpha \\ &= -i_\alpha (-i_\beta (g^\delta \wedge \mathcal{G}_\delta) + \mathcal{G}_\beta) + i_\beta \mathcal{G}_\alpha \\ &= i_\alpha i_\beta (g^\delta \wedge \mathcal{G}_\delta) - i_\alpha \mathcal{G}_\beta + i_\beta \mathcal{G}_\alpha, \end{aligned} \quad (6)$$

we obtain the final form:

$$\omega_{[\alpha\beta]} = i_\beta \mathcal{G}_\alpha - i_\alpha \mathcal{G}_\beta + \frac{1}{2} i_\alpha i_\beta (g^\delta \wedge \mathcal{G}_\delta) - \mathbf{Q}_{\alpha\beta\delta} \wedge g^\delta. \blacksquare$$



Because the non-metricity 1-forms and the torsion 2-forms vanishes in the Levi-Civita connection, the last Lemma applied to the Levi-Civita connection 1-forms  $(\theta^\alpha_\beta)$  yields

$$\theta_{\alpha\beta} = i_\beta dg_\alpha - i_\alpha dg_\beta + \frac{1}{2}i_\alpha i_\beta (g_\gamma \wedge dg^\gamma), \quad (7)$$

(recall that  $\theta_{\alpha\beta} = \theta_{[\alpha\beta]}$  for  $\nabla$ ). The above equation is the correspondent of the Christoffel symbols of the classical tensor calculus, and express the fact that the Levi-Civita connection is completely determined by the cobase  $(g_\alpha)$ .

As a result, we can finally derive our decomposition formula.

**Corollary 15** *The connection 1-forms  $(\omega^\alpha_\beta)$  of an arbitrary connection  $D$  is given in terms of the Levi-Civita connection 1-forms  $(\theta^\alpha_\beta)$ , the non-metricity<sup>4</sup> 1-forms  $(\mathcal{A}^\alpha_\beta)$  and torsion 2-forms  $(\mathbf{T}^\alpha)$  of  $D$  (all relative to a fixed cobase  $(g_\alpha)$ ) via*

$$\omega_{\alpha\beta} = \theta_{\alpha\beta} + \mathcal{A}_{\alpha\beta} - \mathfrak{T}_{\alpha\beta} - \mathbf{Q}_{\alpha\beta\gamma} \wedge g^\gamma,$$

where  $\mathfrak{T}_{\alpha\beta} = \mathfrak{T}_{\alpha\beta}(\mathbf{T}_\gamma) \in \bigwedge^1 M$  is given by

$$\mathfrak{T}_{\alpha\beta} = i_\beta \mathbf{T}_\alpha - i_\alpha \mathbf{T}_\beta + \frac{1}{2}i_\alpha i_\beta (g_\gamma \wedge \mathbf{T}^\gamma),$$

for all  $0 \leq \alpha, \beta \leq \dim(M) - 1$ .

In fact, from the decomposition formula of Lemma 14 and  $\mathcal{G}_\alpha = dg_\alpha - \mathbf{T}_\alpha$ ,

$$\begin{aligned} \omega_{[\alpha\beta]} &= i_\beta dg_\alpha - i_\alpha dg_\beta + \frac{1}{2}i_\alpha i_\beta (g_\gamma \wedge dg^\gamma) \\ &\quad - \left[ i_\beta \mathbf{T}_\alpha - i_\alpha \mathbf{T}_\beta + \frac{1}{2}i_\alpha i_\beta (g_\gamma \wedge \mathbf{T}^\gamma) \right] \\ &\quad - \mathbf{Q}_{\alpha\beta\delta} \wedge g^\delta. \end{aligned}$$

Recognizing that the first sum is just the Levi-Civita connection 1-form  $\theta_{\alpha\beta}$  (see Eq.(7)) and that the second sum is what we called  $\mathfrak{T}_{\alpha\beta}$ , we conclude that  $\omega_{[\alpha\beta]} = \theta_{\alpha\beta} + \mathfrak{T}_{\alpha\beta} - \mathbf{Q}_{\alpha\beta\delta} \wedge g^\delta$ . Now, just recall that  $\omega_{\alpha\beta} = \omega_{[\alpha\beta]} + \mathcal{A}_{\alpha\beta}$ , the sum of its symmetric and anti-symmetric parts. ■

**Remark 16** *In the decomposition formula of  $\omega_{\alpha\beta}$ , the contribution of the torsion derives from the 1-forms  $(\mathfrak{T}_{\alpha\beta})$ , which have the same structure as Eq.(7) for the Levi-Civita connection 1-forms  $(\theta^\alpha_\beta)$ . This symmetry between  $\theta_{\alpha\beta}$  and  $\mathfrak{T}_{\alpha\beta}$  is relevant to the teleparallel formulation of GR. In fact, let  $(g_\alpha)$  be a teleparallel cobase<sup>5</sup> in a teleparallel space with connection  $D$ . Thus  $\mathbf{T}_\alpha = dg_\alpha$ , in which case  $\mathfrak{T}_{\alpha\beta}$  reduces to the right-hand side of Eq.(7). Hence, the torsion 2-forms in the teleparallel space  $(M, g, D)$  are (up to a gauge transformation  $dg_\alpha \mapsto dg_\alpha + d\chi$ ) in a one-to-one correspondence with connection 1-forms in the pseudo-Riemannian space  $(M, g, \nabla)$ . Therefore, if a geometrical theory of gravity can be formulated in one space, it can be in the other [23] [10] [24].*

<sup>4</sup>Recall that  $\mathbf{Q}_{\alpha\beta\gamma}$  are just the components of the non-metricity 2-forms  $\mathbf{Q}_\gamma$ , so that the term " $\mathbf{Q}_{\alpha\beta\gamma} \wedge g^\gamma$ " is another contribution of the non-metricity 1-forms to  $\omega_{\alpha\beta}$ .

<sup>5</sup>That is,  $D_{E_\alpha} E_\beta = 0$  for all  $0 \leq \alpha, \beta \leq 3$ , where  $(E_\alpha)$  is the dual base field of  $(g_\alpha)$ .

We finish this section with another application of Lemma 14, proving a formula useful in the decomposition of the Einstein 1-forms (Eq.(29)).

**Lemma 17**  $d \star (g^\alpha \wedge g^\beta) = \delta g^\beta \wedge \star g^\alpha - \delta g^\alpha \wedge \star g^\beta + g^\beta \wedge \star dg^\alpha - g^\alpha \wedge \star dg^\beta - \star i^\alpha i^\beta (g_\gamma \wedge dg^\gamma)$ .

Indeed, let  $(\theta^\alpha_\beta)$  be the Levi-Civita connection 1-forms of  $(M, \mathbf{g})$  with  $\mathbf{g}$  induced by  $(g_\alpha)$ . So,

$$\begin{aligned} d \star (g^\alpha \wedge g^\beta) &= -\theta^\alpha_\gamma \wedge \star (g^\gamma \wedge g^\beta) - \theta^\beta_\gamma \wedge \star (g^\alpha \wedge g^\gamma) \\ &= \theta^\alpha_\gamma \wedge i^\gamma \star g^\beta - \theta^\beta_\gamma \wedge i^\gamma \star g^\alpha. \end{aligned}$$

Using well-know properties of contraction and Lemma 12,

$$\begin{aligned} d \star (g^\alpha \wedge g^\beta) &= -i^\gamma (\theta^\alpha_\gamma \wedge \star g^\beta) + i^\gamma \theta^\alpha_\gamma \wedge \star g^\beta + i^\gamma (\theta^\beta_\gamma \wedge \star g^\alpha) - i^\gamma \theta^\beta_\gamma \wedge \star g^\alpha \\ &= i^\gamma (\star \theta^\alpha_\gamma \wedge g^\beta) - i^\gamma (\star \theta^\beta_\gamma \wedge g^\alpha) - i^\gamma \theta^\alpha_\gamma \wedge \star g^\beta + i^\gamma \theta^\beta_\gamma \wedge \star g^\alpha \\ &= \star (\theta^\alpha_\gamma \wedge g^\gamma) \wedge g^\beta - \star (\theta^\beta_\gamma \wedge g^\gamma) \wedge g^\alpha - 2 \star \theta^{\alpha\beta} \\ &\quad - i^\gamma \theta^\alpha_\gamma \wedge \star g^\beta + i^\gamma \theta^\beta_\gamma \wedge \star g^\alpha \\ &= \delta g^\beta \wedge \star g^\alpha - \delta g^\alpha \wedge \star g^\beta + g^\alpha \wedge \star dg^\beta - g^\beta \wedge \star dg^\alpha - 2 \star \theta^{\alpha\beta}. \quad (8) \end{aligned}$$

But from Eq.(7),

$$\begin{aligned} \theta^{\alpha\beta} &= i^\beta dg^\alpha - i^\alpha dg^\beta + \frac{1}{2} i^\alpha i^\beta (g_\gamma \wedge dg^\gamma) \\ &= i^\alpha \star (\star dg^\beta) - i^\beta \star (\star dg^\alpha) + \frac{1}{2} i^\alpha i^\beta (g_\gamma \wedge dg^\gamma) \\ &= \star (g^\alpha \wedge \star dg^\beta) - \star (g^\beta \wedge \star dg^\alpha) + \frac{1}{2} i^\alpha i^\beta (g_\gamma \wedge dg^\gamma), \end{aligned}$$

so that

$$2 \star \theta^{\alpha\beta} = 2g^\alpha \wedge \star dg^\beta - 2g^\beta \wedge \star dg^\alpha + \star i^\alpha i^\beta (g_\gamma \wedge dg^\gamma). \quad (9)$$

Thus, from Eqs.(8) and (9),

$$\begin{aligned} d \star (g^\alpha \wedge g^\beta) &= \delta g^\beta \wedge \star g^\alpha - \delta g^\alpha \wedge \star g^\beta + g^\alpha \wedge \star dg^\beta - g^\beta \wedge \star dg^\alpha \\ &\quad + \star i^\alpha i^\beta (g_\gamma \wedge dg^\gamma). \quad \blacksquare \end{aligned}$$

### 2.3 The Non-Metricity Connection $\mathfrak{D}$

**Definition 18** A connection  $\mathfrak{D}$  in a parallelizable manifold  $M$  is a Non-Metricity (NM) connection if and only if  $\mathfrak{D}$  is torsionless and there exists a cobase  $(g_\alpha)$  for which the connection 1-forms  $(\omega^\alpha_\beta)$  of  $\mathfrak{D}$  relative to  $(g_\alpha)$  satisfy  $\omega_{[\alpha\beta]} = 0$ . Or, equivalently, that

$$\omega_{\alpha\beta} = \mathcal{A}_{\alpha\beta}, \text{ for all } 0 \leq \alpha, \beta < \dim(M) - 1,$$

where  $(\mathcal{A}^\alpha_\beta)$  are the non-metricity 1-forms of  $\mathfrak{D}$  relative to  $(g_\alpha)$ . Then  $(g_\alpha)$  is called an adapted cobase of  $\mathfrak{D}$ .

**Example 19** Let  $M$  be a parallelizable manifold,  $(x^\alpha)$  a chart defined on some open neighborhood  $U \subset M$  and  $g = \eta_{\alpha\beta} dx^\alpha \otimes dx^\beta$  a metric induced by the cobase field  $(dx^\alpha)$ . The Levi-Civita connection of  $(M, \mathbf{g})$  is a NM connection, since its connection 1-forms vanishes. This is the trivial NM connection.

**Example 20** Let  $\mathbb{R}^3$  be the Euclidean 3-space, so that if  $(r, \theta, \varphi)$  are polar coordinates in  $U \subset \mathbb{R}^3$ , its metric  $\mathbf{g}|_U = dr \otimes dr + r^2 (d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi)$ . Let  $(g^i)_{0 \leq i \leq 2}$  be the coframe field  $g^0 = dr$ ,  $g^1 = r d\theta$  and  $g^2 = r \sin \theta d\varphi$ . A NM connection  $\mathfrak{D}$  in  $U$  with adapted cobase  $(g^i)$  is defined as follows. Let  $(\mathcal{A}^\alpha_\beta) \in \wedge^1 M$  be such that,

$$\mathcal{A}^1_0 = \frac{1}{r} g^1, \quad \mathcal{A}^1_2 = \frac{1}{r \tan \theta} g^2, \quad \mathcal{A}^2_0 = \frac{1}{r} g^2,$$

and, for arbitrary differentiable functions  $f, g, h : \mathbb{R}^3 \rightarrow \mathbb{R}$ , let  $\mathcal{A}^0_0 = f g^0$ ,  $\mathcal{A}^1_1 = g g^1$  and  $\mathcal{A}^2_2 = h g^2$ . One easily verify that  $dg_i = -\mathcal{A}_{ij} \wedge g^j$  for  $0 \leq i, j \leq 2$ . By declaring  $(\mathcal{A}^\alpha_\beta)$  the non-metricity 1-forms of  $\mathfrak{D}$ ,  $\mathfrak{D}$  is completely defined. So our desired NM connection exists but cannot be unique, given the arbitrariness of the diagonal elements of  $(\mathcal{A}^\alpha_\beta)$ .

**Example 21** Let  $(M, \mathbf{g})$  be a 3-dimensional Riemannian space and  $(x, y, z)$  an orthogonal chart on  $M$  so that  $\mathbf{g} = f^2 dx \otimes dx + g^2 dy \otimes dy + h^2 dz \otimes dz$  for some functions  $f, g, h : M \rightarrow \mathbb{R}$ . Let  $(g^i)_{0 \leq i \leq 2}$  be the coframe field  $g^0 = f dx$ ,  $g^1 = g dy$  and  $g^2 = h dz$ . Define a NM connection  $\mathfrak{D}$  with non-metricity 1-forms  $(\mathcal{A}^\alpha_\beta)$  as follows. Let,

$$\begin{aligned} \mathcal{A}^0_1 &= \frac{f_y}{f g} g^0 + \frac{g_x}{f g} g^1, & \mathcal{A}^0_2 &= \frac{f_z}{f h} g^0 + \frac{h_x}{f h} g^2, \\ \mathcal{A}^1_2 &= \frac{g_z}{g h} g^1 + \frac{h_y}{g h} g^2, \end{aligned}$$

and define  $\mathcal{A}^0_0 = F g^0$ ,  $\mathcal{A}^1_1 = G g^1$  and  $\mathcal{A}^2_2 = H g^2$  for any functions  $F, G, H : M \rightarrow \mathbb{R}$ . So it is easy to prove that  $dg_i = -\mathcal{A}_{ij} \wedge g^j$  for  $0 \leq i, j \leq 2$ , and then  $\mathfrak{D}$  is, up to the diagonal elements of  $(\mathcal{A}^\alpha_\beta)$ , completely defined.

**Remark 22** The obvious generalization of the following example shows that, given any metric on a manifold  $M$ , we can always define locally a NM connection with an adapted cobase field which is a coframe field in  $(M, \mathbf{g})$ .

There exists an identity relating the Levi-Civita connection 1-forms relative to an adapted cobase of a NM connection to the components of the non-metricity 2-forms of that NM connection. This relation is expressed in the following Corollary.

**Corollary 23** Let  $\mathfrak{D}$  be a NM connection on a parallelizable manifold  $M$  and  $\mathbf{Q}_{\alpha\beta\gamma}$  the components of the non-metricity 2-forms  $(\mathbf{Q}_\gamma)$  of  $\mathfrak{D}$  relative to an adapted cobase  $(g_\alpha)$  of  $\mathfrak{D}$ . The connection 1-forms  $(\theta^\alpha_\beta)$  of the Levi-Civita

connection of the pseudo-Riemannian manifold  $(M, \mathbf{g})$  (where  $g = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$ ) satisfy

$$\theta_{\alpha\beta} = \mathbf{Q}_{\alpha\beta\gamma} \wedge g^\gamma.$$

Indeed, by the decomposition formula of Corollary 15,

$$\omega_{[\alpha\beta]} = \omega_{\alpha\beta} - \mathcal{A}_{\alpha\beta} = \theta_{\alpha\beta} - \mathfrak{T}_{\alpha\beta} - \mathbf{Q}_{\alpha\beta\gamma} \wedge g^\gamma.$$

But by the hypothesis on  $\mathfrak{D}$ ,  $\omega_{[\alpha\beta]} = \mathfrak{T}_{\alpha\beta} = 0$ . ■

### 3 Gravitation and Non-Metricity

In section 3.1, we define and utilize the non-metricity gauge to formulate a theory of gravitation, whose field equations are derived from the variational principle. We discuss the form of these field equations when written in the gravitational Lorenz gauge, and deduce a force law for the matter currents coupled to the gravitational field.

Then, in section 3.2, it is shown that the field equations derived in section 3.1 can be expressed completely in terms of the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms of Definition 9. We conclude therefore that the gravitational field may be interpreted as the non-metricity of a flat torsionless connection. Finally, it is exemplified how the non-metricity encodes information about the gravitational field in the Schwarzschild solution.

#### 3.1 Field Equations

The formulation of our theory is based in the following gravitational Lagrangian density, discovered by Wallner [11] and which is (as proven in Appendix A) equivalent to the Einstein-Hilbert Lagrangian density.

**Definition 24** *Let  $M$  be a four-dimensional parallelizable manifold. The Wallner Lagrangian density (WL)  $\mathcal{L} : (\wedge^1 M)^4 \times (\wedge^2 M)^4 \longrightarrow \wedge^4 M$  is*

$$\mathcal{L} = \frac{1}{2} g_\alpha \wedge dg^\beta \wedge \star(g_\beta \wedge dg^\alpha) - \frac{1}{4} g_\alpha \wedge dg^\alpha \wedge \star(g_\beta \wedge dg^\beta),$$

where  $\star : \wedge^p M \longrightarrow \wedge^{4-p} M$  is the Hodge dual relative to the metric  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$  induced by  $(g_\alpha)$  and to the orientation  $\tau = g^0 \wedge g^1 \wedge g^2 \wedge g^3$  of  $M$  [3] [4].

From now on, cobase fields shall be called *gravitational potentials*. Our theory begins with the assumption that the gravitational potentials are cobases adapted to some NM connection  $\mathfrak{D}$ , and their set will be denoted by  $\wedge \mathfrak{D}$ . (By the examples of the last section, we see that  $\wedge \mathfrak{D} \neq \emptyset$ ). This assumption may be referred to as the *non-metricity gauge*. From Remark 22 and the examples shown above, one should see that any gravitational field can indeed be represented by a tetrad in  $\wedge \mathfrak{D}$ .

**Lemma 25** *The restriction  $\mathcal{L}|_{\wedge\mathfrak{D}}$  of the WL to cobases belonging to  $\wedge\mathfrak{D}$  is*

$$\mathcal{L}|_{\wedge\mathfrak{D}} = \frac{1}{2}g_\alpha \wedge dg^\beta \wedge \star(g_\beta \wedge dg^\alpha).$$

Let  $(g_\alpha) \in \wedge\mathfrak{D}$ . So there exists  $(\mathcal{A}_\beta^\alpha) \in \wedge^1 M$  such that  $\mathcal{A}_{\alpha\beta} = \mathcal{A}_{\beta\alpha}$  and  $dg_\alpha = -\mathcal{A}_{\alpha\beta} \wedge g^\beta$ . Thus  $g_\alpha \wedge dg^\alpha = g_\alpha \wedge g_\beta \wedge \mathcal{A}^{\alpha\beta} = 0$ , and only the first term of the WL remains. ■

**Remark 26** *Any field theory whose Lagrangian can be formulated in terms of differential forms is invariant under diffeomorphism transformations [5] [7]. Therefore, the restriction of the WL to gravitational potentials in  $\wedge\mathfrak{D}$  cannot affect the diffeomorphism invariance of the resulting field equations.*

In the following two Lemmas, we determine the variation of the restricted WL. In order to avoid confusion with the codifferential operator  $\delta$ , let's denote by  $\bar{\delta}$  a variation of  $(g_\alpha)$ .

Recall that a variation of  $(g_\alpha)$  imply in a variation of  $\star$ , as the definition of the Hodge dual involves the metric  $\mathbf{g} = \eta_{\alpha\beta}g^\alpha \otimes g^\beta$ . We account for the Hodge variation by means of the following result.

**Lemma 27** *Let  $\omega : \wedge^1 M \rightarrow \wedge^3 M$  be a function of  $(g_\alpha)$ , that is,  $\omega = \omega(g_\alpha)$ . Under a variation  $\bar{\delta}$  of  $(g_\alpha)$ ,*

$$\frac{1}{2}\bar{\delta}(\omega \wedge \star\omega) = \bar{\delta}\omega \wedge \star\omega - \bar{\delta}g_\alpha \wedge \left( i^\alpha \omega \wedge \star\omega + \frac{1}{2} \star(\omega \wedge \star\omega) \wedge \star g^\alpha \right).$$

In fact, first suppose that  $\bar{\delta}\omega = 0$ . So the variation of  $\omega \wedge \star\omega$  derives exclusively from that of the Hodge dual:

$$\bar{\delta}(\omega \wedge \star\omega) = \omega \wedge \bar{\delta}(\star\omega). \quad (10)$$

On the other hand, the identity

$$g^\alpha \wedge g^\beta \wedge g^\gamma \wedge \star\omega = \omega \wedge \star(g^\alpha \wedge g^\beta \wedge g^\gamma)$$

implies that

$$g^\alpha \wedge g^\beta \wedge g^\gamma \wedge \bar{\delta}(\star\omega) = \omega \wedge \bar{\delta}(\star(g^\alpha \wedge g^\beta \wedge g^\gamma)) - \bar{\delta}(g^\alpha \wedge g^\beta \wedge g^\gamma) \wedge \star\omega. \quad (11)$$

The first term above can be written as

$$\bar{\delta}(\star(g^\alpha \wedge g^\beta \wedge g^\gamma)) = \bar{\delta}g_\delta \wedge \varepsilon^{\alpha\beta\gamma\delta} = \bar{\delta}g_\delta \wedge \star(g^\alpha \wedge g^\beta \wedge g^\gamma \wedge g^\delta),$$

while the second as

$$\bar{\delta}(g^\alpha \wedge g^\beta \wedge g^\gamma) = \bar{\delta}g^\alpha \wedge g^\beta \wedge g^\gamma + g^\alpha \wedge \bar{\delta}g^\beta \wedge g^\gamma + g^\alpha \wedge g^\beta \wedge \bar{\delta}g^\gamma.$$

Therefore, Eq.(11) yields

$$\begin{aligned}
& g^\alpha \wedge g^\beta \wedge g^\gamma \wedge \bar{\delta}(\star\omega) \\
&= -\bar{\delta}g_\delta \wedge \omega \wedge \star(g^\alpha \wedge g^\beta \wedge g^\gamma \wedge g^\delta) \\
&- (\bar{\delta}g^\alpha \wedge g^\beta \wedge g^\gamma + g^\alpha \wedge \bar{\delta}g^\beta \wedge g^\gamma + g^\alpha \wedge g^\beta \wedge \bar{\delta}g^\gamma) \wedge \star\omega.
\end{aligned} \tag{12}$$

Now, let  $\omega_{\alpha\beta\gamma} = i_\gamma i_\beta i_\alpha \omega$ . Hence, Eqs.(10) and (12) gives

$$\begin{aligned}
\bar{\delta}(\omega \wedge \star\omega) &= \frac{1}{3!} \omega_{\alpha\beta\gamma} g^\alpha \wedge g^\beta \wedge g^\gamma \wedge \bar{\delta}(\star\omega) \\
&= -\bar{\delta}g_\alpha \wedge \omega \wedge \star(\omega \wedge g^\alpha) - \bar{\delta}g_\alpha \wedge \left( \frac{1}{2} \omega_{\beta\gamma} g^\beta \wedge g^\gamma \right) \wedge \star\omega \\
&= -\bar{\delta}g_\alpha \wedge (\omega \wedge i^\alpha \star\omega + i^\alpha \omega \wedge \star\omega).
\end{aligned} \tag{13}$$

Since

$$\omega \wedge i^\alpha \star\omega = -i^\alpha (\omega \wedge \star\omega) + i^\alpha \omega \wedge \star\omega = -\langle \omega | \omega \rangle \star g^\alpha + i^\alpha \omega \wedge \star\omega,$$

and  $-\langle \omega | \omega \rangle = -i_\omega \star(\star\omega) = -\star(\star\omega \wedge \omega) = \star(\omega \wedge \star\omega)$ , Eq.(13) is equivalent to

$$\frac{1}{2} \bar{\delta}(\omega \wedge \star\omega) = -\bar{\delta}g_\alpha \wedge \left( i^\alpha \omega \wedge \star\omega + \frac{1}{2} \star(\omega \wedge \star\omega) \wedge \star g^\alpha \right).$$

Lastly, by letting  $\omega$  depend on  $(g_\alpha)$ , the total variation becomes

$$\frac{1}{2} \bar{\delta}(\omega \wedge \star\omega) = \bar{\delta}\omega \wedge \star\omega - \bar{\delta}g_\alpha \wedge \left( i^\alpha \omega \wedge \star\omega + \frac{1}{2} \star(\omega \wedge \star\omega) \wedge \star g^\alpha \right). \blacksquare$$

**Lemma 28** *The variation of  $\mathcal{L}|_{\wedge\mathfrak{D}}$  under a variation  $\bar{\delta}$  of  $(g_\alpha) \in \wedge\mathfrak{D}$  is, up to an exact differential,*

$$\bar{\delta}(\mathcal{L}|_{\wedge\mathfrak{D}}) = d(\dots) + \bar{\delta}g_\alpha \wedge \star G^\alpha,$$

where  $(G_\alpha) \in \wedge^1 M$  are the Einstein 1-forms

$$\begin{aligned}
\star G_\alpha &= dg^\beta \wedge \star(g_\beta \wedge dg_\alpha) + d(g_\beta \wedge \star(g_\alpha \wedge dg^\beta)) - \star(\mathcal{L}|_{\wedge\mathfrak{D}}) \wedge \star g_\alpha \\
&- i_\alpha(g_\beta \wedge dg^\gamma) \wedge \star(g_\gamma \wedge dg^\beta).
\end{aligned} \tag{14}$$

Indeed, from Lemmas 25 and 27,

$$\begin{aligned}
\bar{\delta}(\mathcal{L}|_{\wedge\mathfrak{D}}) &= \frac{1}{2} \bar{\delta}(g_\alpha \wedge dg^\beta \wedge \star(g_\beta \wedge dg^\alpha)) \\
&= \bar{\delta}(g_\alpha \wedge dg^\beta) \wedge \star(g_\beta \wedge dg^\alpha) \\
&- \bar{\delta}g_\alpha \wedge (i^\alpha(g_\beta \wedge dg^\gamma) \wedge \star(g_\gamma \wedge dg^\beta) + \star(\mathcal{L}|_{\wedge\mathfrak{D}}) \wedge \star g^\alpha).
\end{aligned} \tag{15}$$

But the first term gives

$$\begin{aligned}
& \bar{\delta} (g_\alpha \wedge dg^\beta) \wedge \star (g_\beta \wedge dg^\alpha) \\
&= \bar{\delta} g_\alpha \wedge dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) + g_\alpha \wedge d\bar{\delta} g^\beta \wedge \star (g_\beta \wedge dg^\alpha) \\
&= \bar{\delta} g_\alpha \wedge dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) + d(\dots) + \bar{\delta} g_\alpha \wedge d(g_\beta \wedge \star (g^\alpha \wedge dg^\beta)) \\
&= d(\dots) + \bar{\delta} g_\alpha \wedge (dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) + d(g_\beta \wedge \star (g^\alpha \wedge dg^\beta))). \tag{16}
\end{aligned}$$

Then, from Eqs.(15) and (16),

$$\begin{aligned}
\bar{\delta} (\mathcal{L}|_{\wedge \mathfrak{D}}) &= d(\dots) \\
&+ \bar{\delta} g_\alpha \wedge [dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) + d(g_\beta \wedge \star (g^\alpha \wedge dg^\beta)) \\
&- i^\alpha (g_\beta \wedge dg^\gamma) \wedge \star (g_\gamma \wedge dg^\beta) - \star (\mathcal{L}|_{\wedge \mathfrak{D}}) \wedge \star g^\alpha].
\end{aligned}$$

The result follows by recognizing the terms between the square brackets as  $\star G^\alpha$  (see Eq.(14)). ■

In the next Lemma, we decompose the Einstein 1-forms in the differential term  $\delta dg_\alpha$  together with the gravitational energy-momentum currents  $\mathcal{T}_\alpha$  (Eq.(18)). In this way, the Einstein equation shall assume the form of four coupled equations with the same structure as the inhomogeneous Maxwell equation, from which we may derive a simple conservation law (Remark 32 and Proposition 33).

**Lemma 29** *The Einstein 1-forms  $(G_\alpha) \in \wedge^1 M$  of the gravitational potentials  $(g_\alpha)$  (Eq.(14)) can be written as*

$$G_\alpha = -\delta dg_\alpha + \mathcal{T}_\alpha, \tag{17}$$

where  $(\mathcal{T}_\alpha) \in \wedge^1 M$  are the gravitational energy-momentum currents:

$$\begin{aligned}
\mathcal{T}_\alpha &= \frac{1}{2} \star (dg_\beta \wedge i_\alpha \star dg^\beta - i_\alpha dg_\beta \wedge \star dg^\beta) \\
&+ \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \wedge g_\alpha + i_\beta d\delta g^\beta \wedge g_\alpha - i_\alpha d\delta g^\beta \wedge g_\beta. \tag{18}
\end{aligned}$$

In fact, the first term of Eq.(14) yields

$$\begin{aligned}
dg^\beta \wedge \star (g_\beta \wedge dg_\alpha) &= dg^\beta \wedge i_\beta \star dg_\alpha \\
&= i_\beta (dg^\beta \wedge \star dg_\alpha) - i_\beta dg^\beta \wedge \star dg_\alpha \\
&= \langle dg_\alpha | dg^\beta \rangle \star g_\beta + \delta g_\beta \wedge g^\beta \wedge \star dg_\alpha, \tag{19}
\end{aligned}$$

where in the last line we used Lemma 13.

Now, the second term of Eq.(14) gives

$$\begin{aligned}
d(g_\beta \wedge \star (g_\alpha \wedge dg^\beta)) &= d(g_\beta \wedge i_\alpha \star dg^\beta) \\
&= d(-i_\alpha (g_\beta \wedge \star dg^\beta) + \star dg_\alpha) \\
&= d\star dg_\alpha - di_\alpha (g_\beta \wedge \star dg^\beta).
\end{aligned}$$

On the other hand,

$$\begin{aligned} g_\beta \wedge \star dg^\beta &= \star^2 (\star dg^\beta \wedge g_\beta) = \star i_\beta \star^2 dg^\beta \\ &= -\star i_\beta dg^\beta = \delta g_\beta \wedge \star g^\beta, \end{aligned}$$

from Lemma 13 again. So,

$$\begin{aligned} d(g_\beta \wedge \star(g_\alpha \wedge dg^\beta)) &= \star^2 (d \star dg_\alpha) + d(\delta g_\beta \wedge \star(g_\alpha \wedge g^\beta)) \\ &= -\star \delta g_\alpha + d\delta g_\beta \wedge \star(g_\alpha \wedge g^\beta) + \delta g_\beta \wedge d \star(g_\alpha \wedge g^\beta). \end{aligned}$$

Therefore, by applying Lemma 17,

$$\begin{aligned} d(g_\beta \wedge \star(g_\alpha \wedge dg^\beta)) &= -\star \delta g_\alpha + d\delta g_\beta \wedge \star(g_\alpha \wedge g^\beta) \\ &\quad + \delta g_\beta \wedge (\delta g^\beta \wedge \star g^\alpha - \delta g^\alpha \wedge \star g^\beta) \\ &\quad + g^\beta \wedge \star dg^\alpha - g^\alpha \wedge \star dg^\beta. \end{aligned} \tag{20}$$

where we used that  $dg_\gamma \wedge g^\gamma = 0$  in the non-metricity gauge.

On the other hand, from Lemma 48 (proved in Appendix A),

$$\begin{aligned} \mathcal{L}|_{\wedge \mathfrak{D}} &= \frac{1}{2} dg_\beta \wedge \star dg^\beta - \frac{1}{2} \delta g_\beta \wedge \star \delta g^\beta \\ &= \left( \frac{1}{2} \langle dg_\beta | dg^\beta \rangle - \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \right) \star 1 \end{aligned}$$

Hence

$$-\star(\mathcal{L}|_{\wedge \mathfrak{D}}) = \frac{1}{2} \langle dg_\beta | dg^\beta \rangle - \frac{1}{2} \delta g_\beta \wedge \delta g^\beta,$$

and the third term of Eq.(14) becomes

$$-\star(\mathcal{L}|_{\wedge \mathfrak{D}}) \wedge \star g_\alpha = \left( \frac{1}{2} \langle dg_\beta | dg^\beta \rangle - \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \right) \wedge \star g_\alpha. \tag{21}$$

Finally, we work on the last term of Eq.(14), for which

$$\begin{aligned} i_\alpha(g_\beta \wedge dg^\gamma) \wedge \star(g_\gamma \wedge dg^\beta) \\ = dg^\gamma \wedge i_\gamma \star dg_\alpha + i_\alpha dg^\gamma \wedge g_\beta \wedge i_\gamma \star dg^\beta. \end{aligned} \tag{22}$$

First,

$$\begin{aligned} dg^\gamma \wedge i_\gamma \star dg_\alpha &= i_\gamma (dg^\gamma \wedge \star dg_\alpha) - i_\gamma dg^\gamma \wedge \star dg_\alpha \\ &= \langle dg_\alpha | dg^\gamma \rangle \star g_\gamma + \delta g_\gamma \wedge g^\gamma \wedge \star dg_\alpha. \end{aligned} \tag{23}$$



Second,

$$\begin{aligned}
& i_\alpha dg^\gamma \wedge g_\beta \wedge i_\gamma \star dg^\beta \\
&= i_\gamma (i_\alpha dg^\gamma \wedge g_\beta \wedge \star dg^\beta) - i_\gamma (i_\alpha dg^\gamma \wedge g_\beta) \wedge \star dg^\beta \\
&= i_\gamma (i_\alpha dg^\gamma \wedge \star \star (\star dg^\beta \wedge g_\beta)) + i_\alpha i_\gamma dg^\gamma \wedge g_\beta \wedge \star dg^\beta + i_\alpha dg^\gamma \wedge \eta_{\beta\gamma} \star dg^\beta \\
&= -i_\gamma (i_\alpha dg^\gamma \wedge \star i_\beta dg^\beta) + i_\alpha i_\gamma dg^\gamma \wedge g_\beta \wedge \star dg^\beta + i_\alpha dg_\beta \wedge \star dg^\beta \\
&= \delta g_\beta \wedge i_\gamma (i_\alpha dg^\gamma \wedge \star g^\beta) - \delta g_\alpha \wedge \star \star (\star dg^\beta \wedge g_\beta) + \langle dg_\beta | dg^\beta \rangle \star g_\alpha \\
&\quad - dg_\beta \wedge i_\alpha \star dg^\beta \\
&= \delta g_\beta \wedge i_\gamma (i_\alpha dg^\gamma \wedge \star g^\beta) - \delta g_\alpha \wedge \delta g_\beta \wedge \star g^\beta + \langle dg_\beta | dg^\beta \rangle \star g_\alpha \\
&\quad - dg_\beta \wedge i_\alpha \star dg^\beta.
\end{aligned} \tag{24}$$

Third,

$$\begin{aligned}
\delta g_\beta \wedge i_\gamma (i_\alpha dg^\gamma \wedge \star g^\beta) &= \delta g_\beta \wedge i_\gamma (g^\beta \wedge \star i_\alpha dg^\gamma) \\
&= \delta g_\beta \wedge \star i_\alpha dg^\beta - \delta g_\beta \wedge g^\beta \wedge i_\gamma \star i_\alpha dg^\gamma \\
&= -\delta g_\beta \wedge \star i_\alpha \star (\star dg^\beta) + \delta g_\beta \wedge g^\beta \wedge i_\gamma \star i_\alpha \star (\star dg^\gamma) \\
&= -\delta g_\beta \wedge \star dg^\beta \wedge g_\alpha + \delta g_\beta \wedge g^\beta \wedge i_\gamma (\star dg^\gamma \wedge g_\alpha) \\
&= -\delta g_\beta \wedge \star dg^\beta \wedge g_\alpha + \delta g_\beta \wedge \star dg_\alpha \wedge g^\beta,
\end{aligned} \tag{25}$$

where the non-metricity gauge has been used again in  $i_\gamma \star dg^\gamma = 0$ . Substituting Eq.(25) in Eq.(24), we obtain

$$\begin{aligned}
& i_\alpha dg^\gamma \wedge g_\beta \wedge i_\gamma \star dg^\beta \\
&= \langle dg_\beta | dg^\beta \rangle \star g_\alpha - dg_\beta \wedge i_\alpha \star dg^\beta \\
&\quad + \delta g_\beta \wedge (\star dg_\alpha \wedge g^\beta - \star dg^\beta \wedge g_\alpha - \delta g_\alpha \wedge \star g^\beta).
\end{aligned} \tag{26}$$

Finally, from Eqs.(22), (23) and (26),

$$\begin{aligned}
& i_\alpha (g_\beta \wedge dg^\gamma) \wedge \star (g_\gamma \wedge dg^\beta) \\
&= \langle dg_\beta | dg^\beta \rangle \star g_\alpha + \langle dg_\alpha | dg^\gamma \rangle \star g_\gamma - dg_\beta \wedge i_\alpha \star dg^\beta \\
&\quad + \delta g_\beta \wedge (2 \star dg_\alpha \wedge g^\beta - \star dg^\beta \wedge g_\alpha - \delta g_\alpha \wedge \star g^\beta).
\end{aligned} \tag{27}$$

Lastly, substituting Eqs.(19), (20), (21) and (27) in Eq.(14), performing some algebraic cancelling and reorganizing the terms, we obtain

$$\begin{aligned}
\star G_\alpha &= -\star d\delta g_\alpha + \left( dg_\beta \wedge i_\alpha \star dg^\beta - \frac{1}{2} \langle dg_\beta | dg^\beta \rangle \star g_\alpha \right) + \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \wedge \star g_\alpha \\
&\quad + d\delta g_\beta \wedge \star (g_\alpha \wedge g^\beta),
\end{aligned}$$

which we recognize as being equivalent to Eq.(17). ■

**Remark 30** The first term of Eq.(18) for the gravitational energy-momentum current  $\mathcal{T}_\alpha$  posses the same structure<sup>6</sup> as the energy-momentum currents of the electromagnetic field  $\mathbf{F} \in \bigwedge^2 M$ , namely [3] [4]

$$\mathcal{T}_\alpha = -\frac{1}{2} \star (\mathbf{F} \wedge i_\alpha \star \mathbf{F} - i_\alpha \mathbf{F} \wedge \star \mathbf{F}).$$

The following terms of  $\mathcal{T}_\alpha$  consists of contributions derived from the codifferential  $\delta g_\alpha$ , all of which vanishes in the gravitational Lorenz gauge, yielding a simple set of equations for the gravitational field (see Remark 35).

**Remark 31** The expression for the Einstein 1-forms given in Lemma 29 can be derived from a decomposition (due to Thirring and Wallner [11] [12] and rediscovered by Sparling [15]) in which

$$\star G_\alpha = d \star \pi_\alpha + \star t_\alpha, \quad (28)$$

where  $(\pi_\alpha) \in \bigwedge^2 M$  are known as the superpotentials and  $(t_\alpha) \in \bigwedge^1 M$  as the pseudo-currents. For the record,  $\pi_\alpha$  and  $t_\alpha$  are given in terms of the Levi-Civita connection 1-forms  $(\theta^\alpha_\beta)$  by

$$\star \pi_\gamma = \frac{1}{2} \theta_{\alpha\beta} \wedge \star (g^\alpha \wedge g^\beta \wedge g_\gamma),$$

$$\star t_\gamma = -\frac{1}{2} \theta_{\alpha\beta} \wedge (\theta_{\gamma\delta} \wedge \star (g^\alpha \wedge g^\beta \wedge g^\delta) + \theta^\beta_\delta \wedge \star (g^\alpha \wedge g^\delta \wedge g_\gamma)),$$

both of which can be derived from the formula of  $G_\alpha$  in terms of the curvature 2-forms  $(\mathcal{R}^\alpha_\beta)$  of the Levi-Civita connection of  $(M, \mathbf{g})$ , i.e. [3] [4]

$$G_\alpha = \frac{1}{2} \mathcal{R}_{\beta\gamma} \wedge \star (g^\beta \wedge g^\gamma \wedge g_\alpha),$$

and from the second Cartan structural equation. By employing Eq.(7), it is possible to express the Thirring-Wallner form (Eq.(28)) just in terms of  $g_\alpha$  and  $dg_\alpha$ , a fact which was one of the motivations leading Wallner to the Lagrangian of Definition 24 [11].

**Remark 32** On the other hand, the idea of expressing the Einstein 1-forms as

$$G_\alpha = -\delta dg_\alpha + \mathcal{T}_\alpha,$$

so that the Einstein equations assumes the formal structure of four inhomogeneous Maxwell equations  $\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha$  coupled to the matter energy-momentum currents  $(\mathcal{J}_\alpha)$ , is due to Rodrigues [17]. From this, we deduce the conservation law

$$\delta (\mathcal{T}_\alpha + \mathcal{J}_\alpha) = 0,$$

which suggests the identification of  $(\mathcal{T}_\alpha)$  as the energy-momentum currents of the gravitational field, instead of the archaic energy-momentum pseudo-tensors usually found in the literature [8] [9]. However, in virtue of the non-metricity gauge adopted here, our Eq.(18) for the gravitational energy-momentum currents is simpler and even more appealing physically than the one appearing in [17].

---

<sup>6</sup>Up to a minus sign.

In the following paragraphs, the variational principle will be finally applied to derive the gravitational field equations. Then we show in Example 35 that, if written in the Lorenz gauge, our field equations becomes a system of coupled Proca equations for the gravitational potentials, whose application to the study of gravitational and electromagnetic waves is briefly outlined in Examples 37 and 38.

To account for the matter energy-momentum currents  $(\mathcal{J}_\alpha) \in \bigwedge^1 M$  (and possibly other classical fields), let  $\mathcal{L}_m : (\bigwedge^1 M)^4 \times (\bigwedge^2 M)^4 \longrightarrow \bigwedge^4 M$  represent the matter Lagrangian. On what follows, it is supposed that under a variation  $\bar{\delta}$  of  $(g_\alpha) \in \bigwedge \mathfrak{D}$ ,

$$\bar{\delta} \mathcal{L}_m = \bar{\delta} g_\alpha \wedge \star \mathcal{J}^\alpha. \quad (29)$$

Details regarding  $\mathcal{L}_m$  and some matter models are described in [3] and [4].

**Proposition 33** *Let  $\mathcal{S} : \bigwedge \mathfrak{D} \longrightarrow \mathbb{R}$  be the action functional*

$$\mathcal{S}[(g_\alpha)] = \int_{\mathcal{V}} (\mathcal{L}|_{\bigwedge \mathfrak{D}} + \mathcal{L}_m)((g_\alpha), (dg_\alpha)),$$

*for  $\mathcal{V} \subset M$  a compact four-dimensional submanifold. Thus  $\mathcal{S}$  is stationary under a variation  $\bar{\delta}$  of  $(g_\alpha) \in \bigwedge \mathfrak{D}$ , that is,*

$$\bar{\delta} \mathcal{S}[(g_\alpha)] = 0,$$

*if and only if the Einstein equations*

$$\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha, \quad (30)$$

*are satisfied.*

In fact, one can prove that [7]

$$\bar{\delta} \mathcal{S} = \bar{\delta} \int_{\mathcal{V}} (\mathcal{L}|_{\bigwedge \mathfrak{D}} + \mathcal{L}_m) = \int_{\mathcal{V}} \bar{\delta} (\mathcal{L}|_{\bigwedge \mathfrak{D}} + \mathcal{L}_m).$$

Hence, by Lemma 28 and Eq.(29),

$$\bar{\delta} \mathcal{S} = \int_{\mathcal{V}} \bar{\delta} g_\alpha \wedge \star (G^\alpha + \mathcal{J}^\alpha) = 0,$$

so that  $G^\alpha + \mathcal{J}^\alpha = 0$ , and the Einstein equations (Eq.(30)) follows from Lemma 29. ■

**Remark 34** *The interaction of the gravitational energy-momentum currents  $(\mathcal{T}_\alpha)$  with the gravitational field is hidden in the old-fashioned geometrical formulation of GR in the pseudo-Riemannian space  $(M, \mathbf{g}, \nabla)$ . Recall that the Einstein equations are given in terms of the Ricci 1-forms  $(\mathcal{R}_\alpha)$  of  $(M, \mathbf{g}, \nabla)$  by*

$$\mathcal{R}_\alpha - \frac{1}{2} \mathcal{R} g_\alpha = \mathcal{J}_\alpha.$$

In this form, only the matter energy-momentum currents  $(\mathcal{J}_\alpha)$  appears in the right-hand side, while the gravitational currents  $(\mathcal{T}_\alpha)$  are disguised in the “geometrical” left-hand side. On the other hand, according to the gravitational equations which were derived above, from where we read

$$\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha,$$

it is seen that both the matter and gravitational energy-momentum currents contributes to and interacts with the gravitational field, realizing the physical idea that the gravitational field interacts with itself.

**Example 35 (Lorenz Gauge)** Lets suppose that  $(g_\alpha)$  satisfy the gravitational Lorenz gauge, for which

$$\delta g_\alpha = 0, \quad 0 \leq \alpha \leq 3.$$

It follows that the gravitational energy-momentum currents  $(\mathcal{T}_\alpha)$  (recall Eq.(18)) becomes

$$\mathcal{T}_\alpha = \frac{1}{2} \star (dg_\beta \wedge i_\alpha \star dg^\beta - i_\alpha dg_\beta \wedge \star dg^\beta),$$

possessing therefore the same structure as the electromagnetic energy-momentum currents (compare with Remark 42). But since

$$\begin{aligned} i_\alpha dg_\beta \wedge \star dg^\beta &= i_\alpha (dg_\beta \wedge \star dg^\beta) - dg_\beta \wedge i_\alpha \star dg^\beta \\ &= \langle dg_\beta | dg^\beta \rangle \star g_\alpha - dg_\beta \wedge \star (g_\alpha \wedge dg^\beta), \end{aligned} \quad (31)$$

the currents  $(\mathcal{T}_\alpha)$  can be rewritten as

$$\mathcal{T}_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) - \frac{1}{2} \langle dg_\beta | dg^\beta \rangle g_\alpha. \quad (32)$$

On the other hand, recall that the Laplace-Beltrami operator is defined by  $\square = d\delta + \delta d$  (see [3] or [4]), so that in the Lorenz gauge,

$$\square g_\alpha = (d\delta + \delta d) g_\alpha = \delta dg_\alpha. \quad (33)$$

Finally, from Eqs.(32) and (33) together with the Einstein equation (Eq.(30)), our gravitational field equations becomes the following system of four coupled Proca equations,

$$\square g_\alpha + \frac{1}{2} \langle dg_\beta | dg^\beta \rangle g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) + \mathcal{J}_\alpha, \quad (34)$$

whose variable mass  $\mathfrak{M} = \frac{1}{2} \langle dg_\beta | dg^\beta \rangle$  derives entirely from the gravitational field, while the sources are both from gravitational and matter origins, namely,  $i_{dg_\beta} (g_\alpha \wedge dg^\beta)$  and  $\mathcal{J}_\alpha$  respectively.

**Remark 36** The variable mass  $\mathfrak{M}$  appearing in the Proca equations for the gravitational potentials in the Lorenz gauge (Eq.(34)) cannot be identified as the graviton mass. In fact, if the mass term  $\frac{1}{4} m^2 g_\alpha \wedge \star g^\alpha$  is included in the WL,

it can be proven that the currents appearing in our field equations (Eq.(30)) receives an additional mass term, becoming

$$\delta dg_\alpha = \mathcal{T}_\alpha + m^2 g_\alpha + \mathcal{J}_\alpha. \quad (35)$$

This, in turn, adds a correction to the variable mass  $\mathfrak{M}$  of our Proca equations, which now reads

$$\square g_\alpha + \left( \frac{1}{2} \langle dg_\beta | dg^\beta \rangle + m^2 \right) g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) + \mathcal{J}_\alpha.$$

Alternatively, if we look at the geometrical formulation of GR in the pseudo-Riemannian space  $(M, \mathbf{g}, \nabla)$ , the field equations after the inclusion of the mass term  $\frac{1}{4}m^2 g_\alpha \wedge \star g^\alpha$  are given by

$$\mathcal{R}_\alpha - \frac{1}{2} \mathcal{R} g_\alpha + m^2 g_\alpha = \mathcal{J}_\alpha.$$

The presence of a mass term in the WL, therefore, is equivalent to the introduction of a cosmological constant. So, while the variable mass  $\mathfrak{M}$  of Eq.(34) is effective in character, depending on the configuration of the gravitational field, the mass  $m$  (or, equivalently, a cosmological constant) interacts with the gravitational potential by means of a contribution to the gravitational energy-momentum currents, as seen in Eq.(35).

**Example 37** It is interesting to note that our vacuum ( $\mathcal{J}_\alpha = 0$ ) gravitational equations in the Lorenz gauge (Eq.(34)) are given by

$$\square g_\alpha + \frac{1}{2} \langle dg_\beta | dg^\beta \rangle g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta).$$

Therefore, even in the absence of matter, the Proca equations describing the propagation of the gravitational potentials posses a source, but which is now purely gravitational in origin. This is a statement of the non-linearity inherent to the propagation of the gravitational field, and a mathematical realization of the physical picture that gravitational disturbances may itself be a source of gravitational fields.

**Example 38** In electrovacuum, the only contribution to the “matter” energy-momentum currents ( $\mathcal{J}_\alpha$ ) derives from the electromagnetic field, for which (recall Remark 30)

$$\mathcal{J}_\alpha = \frac{1}{2} \star (i_\alpha \mathbf{F} \wedge \star \mathbf{F} - \mathbf{F} \wedge i_\alpha \star \mathbf{F}) = \frac{1}{2} \langle \mathbf{F} | \mathbf{F} \rangle g_\alpha - i_\mathbf{F} (\mathbf{F} \wedge g_\alpha).$$

Accordingly, our gravitational field equations in the Lorenz gauge (Eq.(34)) becomes

$$\square g_\alpha + \frac{1}{2} (\langle dg_\beta | dg^\beta \rangle - \langle \mathbf{F} | \mathbf{F} \rangle) g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) - i_\mathbf{F} (\mathbf{F} \wedge g_\alpha).$$

We then see that the existence of a nonvanishing electromagnetic field changes the effective mass of propagation of the gravitational potentials and introduces a contribution to the source term in the above Proca equations. This result suggests that the existence of electromagnetic oscillations may be a source for gravitational perturbation, which is consonant with many solutions of the Einstein-Maxwell equations (and its linearized version) which describes a coupled system of gravitational-electromagnetic waves (some of which are [18], [19] and [20]).

Our form of the Einstein equation admits a very simple “force law”, which is just the analog of the Newtonian theorem of work and energy variation.

**Definition 39** Let  $\xi \in \sec TM$  be a Killing vector field. The energy-flow of the gravitational potentials  $(g_\alpha)$  along  $\xi$  is the 1-form  $\mathcal{W}_\xi \in \bigwedge^1 M$  given by

$$\mathcal{W}_\xi = \frac{1}{2} \star (dg_\beta \wedge i_\xi \star dg^\beta - i_\xi dg_\beta \wedge \star dg^\beta),$$

while  $\delta \mathcal{W}_\xi = \mathcal{F}_\xi$  is the gravitational force along  $\xi$ .

**Remark 40** Suppose that the dual  $E_\alpha$  of a gravitational potential  $g_\alpha$  turns out to be a Killing vector field. So the energy-flow  $\mathcal{W}_\xi$  of the gravitational potentials along  $\xi = E_\alpha$  is just the first term of the gravitational energy-momentum currents of Eq.(18). In fact, if the Lorenz gauge  $\delta g_\beta = 0$  ( $0 \leq \beta \leq 3$ ) is assumed,  $\mathcal{W}_\xi = \mathcal{T}_\alpha$ .

**Lemma 41** Let  $(g_\alpha)$  be gravitational potentials obeying Eq.(30),  $\xi \in \sec TM$  a Killing vector field and  $\mathcal{W}_\xi$  the energy-flow of  $(g_\alpha)$  along  $\xi$ . So

$$\mathcal{F}_\xi = \delta \mathcal{W}_\xi = \langle i_\xi dg_\alpha | \mathcal{T}_\alpha + \mathcal{J}_\alpha \rangle,$$

where  $(\mathcal{T}_\alpha)$  and  $(\mathcal{J}_\alpha)$  are the gravitational and matter energy-momentum currents, respectively.

Indeed, denoting by  $L_\xi$  the Lie derivative along  $\xi$  and applying the Cartan’s formula  $L_\xi = di_\xi + i_\xi d$ ,

$$\begin{aligned} d \star \mathcal{W}_\xi &= \frac{1}{2} di_\xi (dg_\alpha \wedge \star dg^\alpha) - dg_\alpha \wedge di_\xi \star dg^\alpha \\ &= \frac{1}{2} L_\xi (dg_\alpha \wedge \star dg^\alpha) - dg_\alpha \wedge di_\xi \star dg^\alpha \\ &= dg_\alpha \wedge L_\xi \star dg^\alpha - dg_\alpha \wedge di_\xi \star dg^\alpha \\ &= dg_\alpha \wedge i_\xi d \star dg^\alpha \\ &= -dg_\alpha \wedge i_\xi \star \delta dg^\alpha. \end{aligned}$$

From Einstein equation,

$$\begin{aligned} d \star \mathcal{W}_\xi &= -dg_\alpha \wedge i_\xi \star (\mathcal{T}_\alpha + \mathcal{J}_\alpha) \\ &= i_\xi dg_\alpha \wedge \star (\mathcal{T}_\alpha + \mathcal{J}_\alpha), \end{aligned}$$

so that

$$\delta \mathcal{W}_\xi = - \star (i_\xi dg_\alpha \wedge \star (\mathcal{T}_\alpha + \mathcal{J}_\alpha)) = \langle i_\xi dg_\alpha | \mathcal{T}^\alpha + \mathcal{J}^\alpha \rangle. \blacksquare$$

**Remark 42** The above force law for the gravitational interaction with matter currents is analogous to the Lorentz force of electrodynamics [3]. Indeed, let  $\mathbf{F} \in \bigwedge^2 M$  be an electromagnetic field interacting with the current  $\mathcal{J} \in \bigwedge^1 M$  according to the Maxwell inhomogeneous equation  $\delta \mathbf{F} = \mathcal{J}$ . Following the same steps of the proof of the above Lemma, one can show that  $\mathcal{J}$  obeys the force law

$$\delta T_\xi = \langle i_\xi \mathbf{F} | \mathcal{J} \rangle.$$

Here,  $T_\xi$  is the electromagnetic energy-momentum current along  $\xi$ , given by

$$T_\xi = -\frac{1}{2} \star (\mathbf{F} \wedge i_\xi \star \mathbf{F} - i_\xi \mathbf{F} \wedge \star \mathbf{F}).$$

Compare  $T_\xi$  with the discussion of Remark 30.

### 3.2 Gravitation as Non-Metricity

On what follows, we rewrite our field equations (Eq.(30)) entirely in terms of the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms of the NM connection  $\mathfrak{D}$  to which our gravitational potentials  $(g_\alpha)$  are adapted, together with the matter energy-momentum currents. This will show that the gravitational field can be interpreted as the manifestation of the non-metricity of  $\mathfrak{D}$ .

**Lemma 43** Let  $(g_\alpha) \in \bigwedge \mathfrak{D}$  and  $\mathbf{Q}_{\alpha\beta\gamma}$  be the components of the non-metricity 2-forms  $(\mathbf{Q}_\gamma)$  of  $\mathfrak{D}$  relative to  $(g_\alpha)$ . So,

$$i_\nu (\delta dg_\mu) = \mathbf{Q}_{\mu[\alpha\nu];}{}^\alpha - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_\nu^{[\alpha\beta]} - \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_\beta^{\alpha\beta}. \quad (36)$$

Also, let  $(\mathcal{T}_\alpha) \in \bigwedge^1 M$  be the gravitational energy-momentum currents (Eq.(18)) of the gravitational potentials  $(g_\alpha)$ . Hence,

$$\begin{aligned} i_\nu \mathcal{T}_\mu &= \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu} - \mathbf{Q}_{\alpha\nu}{}^\alpha{}_{;\mu} \\ &+ \left[ \mathbf{Q}_{\alpha\beta}{}^{\alpha;\beta} + \frac{1}{2} \left( \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]} + \mathbf{Q}_{\alpha\beta}{}^\alpha \mathbf{Q}_\gamma^{\beta\gamma} \right) \right] \eta_{\mu\nu}. \end{aligned} \quad (37)$$

First we prove Eq.(36). Let  $(\theta_\beta)$  be the Levi-Civita connection 1-forms of the pseudo-Riemannian space  $(M, \mathbf{g})$ , where  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$ . By Corollary 23,

$$dg_\mu = -\theta_{\mu\alpha} \wedge g^\alpha = -(\mathbf{Q}_{\mu\alpha\beta} g^\beta) \wedge g^\alpha = \mathbf{Q}_{\mu\alpha\beta} g^\alpha \wedge g^\beta. \quad (38)$$

So,

$$\begin{aligned} \star d \star (dg_\mu) &= \star d (\mathbf{Q}_{\mu\alpha\beta} \star (g^\alpha \wedge g^\beta)) \\ &= \star (d \mathbf{Q}_{\mu\alpha\beta} \wedge \star (g^\alpha \wedge g^\beta)) + \mathbf{Q}_{\mu\alpha\beta} \star d \star (g^\alpha \wedge g^\beta) \\ &= \mathbf{Q}_{\mu\alpha\beta;\delta} \star (g^\delta \wedge \star (g^\alpha \wedge g^\beta)) + \mathbf{Q}_{\mu\alpha\beta} \star d \star (g^\alpha \wedge g^\beta). \end{aligned} \quad (39)$$

On one hand,

$$\star (g^\delta \wedge \star (g^\alpha \wedge g^\beta)) = -i^\delta (g^\alpha \wedge g^\beta) = \eta^{\beta\delta} g^\alpha - \eta^{\alpha\delta} g^\beta,$$

and therefore

$$\begin{aligned}
\mathbf{Q}_{\mu\alpha\beta;\delta} \star (g^\delta \wedge \star (g^\alpha \wedge g^\beta)) &= \mathbf{Q}_{\mu\alpha\beta;\delta} \eta^{\beta\delta} g^\alpha - \mathbf{Q}_{\mu\alpha\beta;\delta} \eta^{\alpha\delta} g^\beta \\
&= \mathbf{Q}_{\mu\alpha\beta;\delta} g^\alpha - \mathbf{Q}_{\mu\alpha\beta;\delta} g^\beta \\
&= (\mathbf{Q}_{\mu\alpha\beta;\delta} - \mathbf{Q}_{\mu\beta\alpha;\delta}) g^\alpha \\
&= \mathbf{Q}_{\mu[\alpha\beta];\delta} g^\alpha.
\end{aligned} \tag{40}$$

Now, on the other hand,

$$\begin{aligned}
d \star (g^\alpha \wedge g^\beta) &= -\theta^\alpha_\gamma \wedge \star (g^\gamma \wedge g^\beta) - \theta^\beta_\gamma \wedge \star (g^\alpha \wedge g^\gamma) \\
&= -\mathbf{Q}_{\gamma\delta} g^\delta \wedge \star (g^\gamma \wedge g^\beta) - \mathbf{Q}_{\gamma\delta} g^\delta \wedge \star (g^\alpha \wedge g^\gamma),
\end{aligned}$$

using Corollary 23 again. Thus

$$\star d \star (g^\alpha \wedge g^\beta) = -\mathbf{Q}_{\gamma\delta}^\alpha \star (\star (g^\gamma \wedge g^\beta) \wedge g^\delta) - \mathbf{Q}_{\gamma\delta}^\beta \star (\star (g^\alpha \wedge g^\gamma) \wedge g^\delta).$$

But since

$$\begin{aligned}
\star (\star (g^\gamma \wedge g^\beta) \wedge g^\delta) &= -i^\delta (g^\gamma \wedge g^\beta) = \eta^{\beta\delta} g^\gamma - \eta^{\gamma\delta} g^\beta, \\
\star (\star (g^\alpha \wedge g^\gamma) \wedge g^\delta) &= -i^\delta (g^\alpha \wedge g^\gamma) = \eta^{\gamma\delta} g^\alpha - \eta^{\alpha\delta} g^\gamma,
\end{aligned}$$

we obtain:

$$\begin{aligned}
\star d \star (g^\alpha \wedge g^\beta) &= -\mathbf{Q}_{\gamma\delta}^\alpha (\eta^{\beta\delta} g^\gamma - \eta^{\gamma\delta} g^\beta) - \mathbf{Q}_{\gamma\delta}^\beta (\eta^{\gamma\delta} g^\alpha - \eta^{\alpha\delta} g^\gamma) \\
&= \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta - \mathbf{Q}_{\gamma}^{\beta\gamma} g^\alpha + (\mathbf{Q}_{\gamma}^{\beta\alpha} - \mathbf{Q}_{\gamma}^{\alpha\beta}) g^\gamma \\
&= \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta - \mathbf{Q}_{\gamma}^{\beta\gamma} g^\alpha + (\mathbf{Q}_{\gamma}^{\alpha\beta} - \mathbf{Q}_{\gamma}^{\beta\alpha}) g^\gamma \\
&= \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta - \mathbf{Q}_{\gamma}^{\beta\gamma} g^\alpha + \mathbf{Q}_{\gamma}^{[\alpha\beta]} g^\gamma.
\end{aligned}$$

So, multiplying by  $\mathbf{Q}_{\mu\alpha\beta}$  yields

$$\begin{aligned}
\mathbf{Q}_{\mu\alpha\beta} \star d \star (g^\alpha \wedge g^\beta) &= \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{[\alpha\beta]} g^\gamma + \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{\beta\gamma} g^\alpha \\
&= \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{[\alpha\beta]} g^\gamma + (\mathbf{Q}_{\mu\alpha\beta} - \mathbf{Q}_{\mu\beta\alpha}) \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta \\
&= \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{[\alpha\beta]} g^\gamma + \mathbf{Q}_{\mu[\alpha\beta]} \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta.
\end{aligned} \tag{41}$$

Hence, from Eqs.(39), (40) and (41), we deduce:

$$\star d \star (dg_\mu) = \mathbf{Q}_{\mu[\alpha\beta];\gamma}^\beta g^\alpha + \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\gamma}^{[\alpha\beta]} g^\gamma + \mathbf{Q}_{\mu[\alpha\beta]} \mathbf{Q}_{\gamma}^{\alpha\gamma} g^\beta.$$

By contracting with  $i_\nu$ ,

$$i_\nu (\star d \star (dg_\mu)) = \mathbf{Q}_{\mu[\nu\beta];\gamma}^\beta + \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\nu}^{[\alpha\beta]} + \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_{\gamma}^{\alpha\gamma}.$$

Finally, since  $i_\nu (\delta dg_\mu) = -i_\nu (\star d \star (dg_\mu))$ , we obtain

$$i_\nu (\delta dg_\mu) = \mathbf{Q}_{\mu[\alpha\nu];\gamma}^\alpha - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_{\nu}^{[\alpha\beta]} - \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_{\beta}^{\alpha\beta}.$$



Now we derive Eq.(37). Using that

$$dg_\alpha \wedge i_\mu \star dg^\alpha = \langle dg_\alpha | dg^\alpha \rangle \star g_\mu - i_\mu dg_\alpha \wedge \star dg^\alpha,$$

the gravitational energy-momentum currents (Eq.(18)) can be written as

$$\begin{aligned} \mathcal{T}_\mu &= \frac{1}{2} \langle dg_\alpha | dg^\alpha \rangle g_\mu + i_{i_\mu dg_\alpha} dg^\alpha \\ &\quad + \frac{1}{2} \delta g_\alpha \wedge \delta g^\alpha \wedge g_\mu + i_\alpha d\delta g^\alpha \wedge g_\mu - i_\mu d\delta g^\alpha \wedge g_\alpha. \end{aligned}$$

Contraction with  $i_\nu$  yields

$$\begin{aligned} i_\nu \mathcal{T}_\mu &= \left( \frac{1}{2} \langle dg_\alpha | dg^\alpha \rangle + \frac{1}{2} \delta g_\alpha \wedge \delta g^\alpha + i_\alpha d\delta g^\alpha \right) \eta_{\mu\nu} \\ &\quad + i_\nu (i_{i_\mu dg_\alpha} dg^\alpha) - i_\mu d\delta g_\nu. \end{aligned} \quad (42)$$

First, use Eq.(38) to obtain

$$\langle dg_\alpha | dg^\alpha \rangle = \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}_{\delta\epsilon}^\alpha \langle g^\beta \wedge g^\gamma | g^\delta \wedge g^\epsilon \rangle.$$

But

$$\begin{aligned} \langle g^\beta \wedge g^\gamma | g^\delta \wedge g^\epsilon \rangle &= i^\gamma i^\beta (g^\delta \wedge g^\epsilon) \\ &= i^\gamma (\eta^{\beta\delta} g^\epsilon - \eta^{\beta\epsilon} g^\delta) = \eta^{\beta\delta} \eta^{\gamma\epsilon} - \eta^{\beta\epsilon} \eta^{\gamma\delta} \end{aligned}$$

implies

$$\begin{aligned} \langle dg_\alpha | dg^\alpha \rangle &= \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}_{\delta\epsilon}^\alpha \eta^{\beta\delta} \eta^{\gamma\epsilon} - \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}_{\delta\epsilon}^\alpha \eta^{\beta\epsilon} \eta^{\gamma\delta} \\ &= \mathbf{Q}_{\alpha\beta\gamma} (\mathbf{Q}^{\alpha\beta\gamma} - \mathbf{Q}^{\alpha\gamma\beta}) = \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]}. \end{aligned} \quad (43)$$

Second, from Lemma 12 and Corollary 23,

$$\delta g_\alpha = i^\beta \theta_{\beta\alpha} = i^\beta (\mathbf{Q}_{\beta\alpha\gamma} g^\gamma) = \mathbf{Q}_{\beta\alpha}^\beta,$$

so that

$$\delta g_\alpha \wedge \delta g^\alpha = \mathbf{Q}_{\beta\alpha}^\beta \mathbf{Q}_\gamma^{\alpha\gamma}. \quad (44)$$

Third,

$$i^\alpha d\delta g_\alpha = i^\alpha (\mathbf{Q}_{\beta\alpha}^\beta{}_{;\gamma} g^\gamma) = \mathbf{Q}_{\alpha\beta}^\alpha{}^\beta{}_{;\gamma} g^\gamma \quad (45)$$

and

$$i_\mu d\delta g_\nu = i_\mu (\mathbf{Q}_{\beta\nu}^\beta{}_{;\gamma} g^\gamma) = \mathbf{Q}_{\alpha\nu}^\alpha{}_{;\mu} g^\mu. \quad (46)$$

Lastly, from Eq.(38),

$$\begin{aligned} i_\mu dg_\alpha &= i_\mu (\mathbf{Q}_{\alpha\beta\gamma} g^\beta \wedge g^\gamma) = \mathbf{Q}_{\alpha\beta\gamma} (\delta_\mu^\beta g^\gamma - \delta_\mu^\gamma g^\beta) \\ &= (\mathbf{Q}_{\alpha\mu\beta} - \mathbf{Q}_{\alpha\beta\mu}) g^\beta = \mathbf{Q}_{\alpha[\mu\beta]} g^\beta, \end{aligned}$$

which imply

$$\begin{aligned}
i_{i_\mu dg_\alpha} dg^\alpha &= \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}_{\gamma\delta}^\alpha i^\beta (g^\gamma \wedge g^\delta) \\
&= \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}_{\gamma\delta}^\alpha (\eta^{\beta\gamma} g^\delta - \eta^{\beta\delta} g^\gamma) \\
&= \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} g_\gamma.
\end{aligned}$$

Hence:

$$i_\nu (i_{i_\mu dg_\alpha} dg^\alpha) = \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu}. \quad (47)$$

Finally, substituting Eqs.(43) to (47) in Eq.(42), we conclude:

$$i_\nu \mathcal{T}_\mu = \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu} - \mathbf{Q}_{\alpha\nu}{}^\alpha{}_{;\mu} \quad (48)$$

$$+ \left[ \mathbf{Q}_{\alpha\beta}{}^\alpha{}^\beta{}_{;\mu} + \frac{1}{2} \left( \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]} + \mathbf{Q}_{\beta\alpha}{}^\beta{}^\alpha \mathbf{Q}_\gamma{}^{\alpha\gamma} \right) \right] \eta_{\mu\nu}. \blacksquare \quad (49)$$

**Proposition 44** *The Einstein equations (Eq.(30)) for the gravitational potentials  $(g_\alpha) \in \bigwedge^2 \mathfrak{D}$  coupled to the matter energy-momentum currents  $(\mathcal{J}_\alpha)$  are equivalent to the following field equations for the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms  $(\mathbf{Q}_\gamma)$  of  $\mathfrak{D}$  relative to  $(g_\alpha)$ ,*

$$\begin{aligned}
i_\nu \mathcal{T}_\mu &= \mathbf{Q}_{\mu[\alpha\nu]}{}^\alpha + \mathbf{Q}_{\alpha\nu}{}^\alpha{}_{;\mu} - \left[ \mathbf{Q}_{\alpha\beta}{}^{\alpha;\beta} + \frac{1}{2} \left( \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]} + \mathbf{Q}_{\alpha\beta}{}^\alpha \mathbf{Q}_\gamma{}^{\beta\gamma} \right) \right] \eta_{\mu\nu} \\
&\quad - \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu} - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_\nu{}^{[\alpha\beta]} - \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_\beta{}^{\alpha\beta}. \quad (50)
\end{aligned}$$

The result follows from a direct substitution of Eqs.(36) and (37) in the Einstein equations (Eq.(30)).  $\blacksquare$

**Remark 45** *The linearized gravitational field equations in terms of the components  $\mathbf{Q}_{\alpha\beta\gamma}$  of the non-metricity 2-forms are*

$$i_\nu \mathcal{T}_\mu = \mathbf{Q}_{\mu[\alpha\nu]}{}^\alpha + \mathbf{Q}_{\alpha\nu}{}^\alpha{}_{;\mu} - \mathbf{Q}_{\alpha\beta}{}^{\alpha;\beta},$$

as seen easily from Eq.(50). They constitute a coupled system of first order partial differential equations.

**Example 46 (Schwarzschild solution)** *Let  $m > 0$ ,  $M = \mathbb{R} \times ]2m, \infty[ \times S^2$  and  $(t, r, \theta, \varphi)$  be the natural coordinates of  $M$ . Also, let*

$$\mathbf{g} = - \left( 1 - \frac{2m}{r} \right) dt \otimes dt + \frac{1}{1 - 2m/r} dr \otimes dr + r^2 \omega,$$

where  $\omega$  is the pull-back of the Euclidean metric of  $S^2$ ,

$$\omega = d\theta \otimes d\theta + \sin^2 \theta d\varphi \otimes d\varphi.$$

Then  $(M, \mathbf{g})$  is called the Schwarzschild solution with mass  $m$ , and  $(t, r, \theta, \varphi)$  are known as Schwarzschild coordinates [22]. The gravitational potentials of  $(M, \mathbf{g})$  can be described by the coframe field  $(g_\alpha)$  such that

$$g^0 = \sqrt{1 - \frac{2m}{r}} dt, \quad g^1 = \frac{1}{\sqrt{1 - 2m/r}} dr, \quad g^2 = r d\theta, \quad g^3 = r \sin \theta d\varphi.$$

A NM connection can be defined for which  $(g_\alpha)$  is an adapted coframe field, once we choose its non-metricity 1-forms  $(\mathcal{A}_\beta^\alpha)$  as

$$\mathcal{A}_1^0 = \frac{1}{\sqrt{1-2m/r}} \frac{m}{r^2} g^0, \quad \mathcal{A}_1^2 = \frac{1}{r} \sqrt{1-\frac{2m}{r}} g^2,$$

$$\mathcal{A}_1^3 = \frac{1}{r} \sqrt{1-\frac{2m}{r}} g^3, \quad \mathcal{A}_2^3 = \frac{1}{r \tan \theta} g^3,$$

and with diagonal elements

$$\mathcal{A}_0^0 = F_0 g^0, \dots, \mathcal{A}_3^3 = F_3 g^3,$$

where  $(F_\alpha)$  are any differentiable functions  $M \rightarrow \mathbb{R}$ . It is easily proven that

$$dg_\alpha = -\mathcal{A}_{\alpha\beta} \wedge g^\alpha,$$

so that our NM is well-defined.

**Remark 47** A discussion of the Schwarzschild solution in terms of non-metricity is also presented by Notte-Cuello, da Rocha and Rodrigues in [21]. However, instead of using a non-metricity gauge or NM connections, these authors considered the situation in which the gravitational field is derived from the non-metricity of the Levi-Civita connection compatible with a Minkowski metric, defined over the Schwarzschild spacetime. Therefore, their work requires a bi-metric theory of gravitation, while our theory only requires a cobase satisfying the non-metricity gauge.

## 4 Discussion

In trying to better understand the gauge nature of gravitation, Thirring and Wallner [3] [11] [12] were led to a gravitational Lagrangian density involving only the cobase that represents the gravitational potentials (recall Definition 24 and Remark 31). Their approach, as the reader may be convinced by studying our Appendix A, is not restricted to any geometrical interpretation of the gravitational field. The latter is then realized as a *legitimate* field living in the spacetime manifold, in a sense similar to that with which Faraday, Maxwell and Lorentz attributed to the electromagnetism.

Now, concerned with the existence of conservation laws in GR and guided by Thirring and Wallner writings, Rodrigues and his collaborators [5] [6] [16] [17] proposed to rewrite the Einstein equations coupled to the matter currents  $(\mathcal{J}_\alpha)$  as

$$\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha,$$

where the 1-forms  $(\mathcal{T}_\alpha)$  are identified with the gravitational energy-momentum currents. This parallels the equations of gravitation with the Maxwell inhomogeneous equation of electrodynamics, and realizes the physical idea that the

gravitational energy-momentum currents are itself a source for gravitational fields.

On the other hand, many authors have been concerned with new geometrical interpretations of GR, principally in teleparallel spaces. In particular, Nester [25], Adak and his collaborators [26]–[29] studied the geometrical formulation of GR based on a flat torsionless connection, where the gravitational field is manifest in the non-metricity of such a connection. This approach, which we have called the “non-metricity formulation of GR”, has been referred by the latter authors (starting with Nester) as the “Symmetric Teleparallel General Relativity” (STGR).

In the present paper, we have unified the works of Thirring, Wallner, Rodrigues, Nester, Adak and many others. We begun with the gravitational formalism of Thirring and Wallner, conceiving the gravitational potentials as a cobase field ( $g_\alpha$ ) living in a parallelizable spacetime manifold. Then, from our discussion of the NM connections, we introduced the notion of the non-metricity gauge, for which the gravitational potentials satisfy

$$g_\alpha \wedge dg^\alpha = 0.$$

The geometrical interpretation of this gauge is that the cobase ( $g_\alpha$ ) is *adapted* to a given NM connection  $\mathfrak{D}$ , which means that the connection 1-forms of  $\mathfrak{D}$  coincides with its non-metricity 1-forms relative to ( $g_\alpha$ ).

Then the Wallner Lagrangian density, when restricted to the non-metricity gauge, becomes

$$\mathcal{L}|_{\wedge\mathfrak{D}} = \frac{1}{2}g_\alpha \wedge dg^\beta \wedge \star(g_\beta \wedge dg^\alpha).$$

By employing the variational principle and following Rodrigues remarks, we obtained the gravitational field equations coupled to the matter energy-momentum currents ( $\mathcal{J}_\alpha$ ) as  $\delta dg_\alpha = \mathcal{T}_\alpha + \mathcal{J}_\alpha$ . However, from our concern with the non-metricity formulation of gravitation, we found that the gravitational energy-momentum currents in the non-metricity gauge assumes a particularly simple and physically appealing form, namely,

$$\begin{aligned} \mathcal{T}_\alpha &= \frac{1}{2} \star (dg_\beta \wedge i_\alpha \star dg^\beta - i_\alpha dg_\beta \wedge \star dg^\beta) \\ &+ \frac{1}{2} \delta g_\beta \wedge \delta g^\beta \wedge g_\alpha + i_\beta d\delta g^\beta \wedge g_\alpha - i_\alpha d\delta g^\beta \wedge g_\beta, \end{aligned}$$

principally if compared with Rodrigues’ original expression for ( $\mathcal{T}_\alpha$ ) [17].

From this, we could deduce that if the gravitational Lorenz gauge is assumed, for which

$$\delta g_\alpha = 0, \quad 0 \leq \alpha \leq 3,$$

the gravitational field equations becomes a system of four coupled Proca equations with variable mass,

$$\square g_\alpha + \frac{1}{2} \langle dg_\beta | dg^\beta \rangle g_\alpha = i_{dg_\beta} (g_\alpha \wedge dg^\beta) + \mathcal{J}_\alpha.$$

As we have indicated in Examples 37 and 38, these equations may be of interest in the study of the propagation of gravitational-electromagnetic waves, as illustrated in the solutions of [18], [19] and [20].

As another consequence of our study of the gravitational equations, we proved a particularly simple force law for the matter currents coupled to the gravitational field. Namely, that if we identify the 1-form

$$\mathcal{W}_\xi = \frac{1}{2} \star (dg_\beta \wedge i_\xi \star dg^\beta - i_\xi dg_\beta \wedge \star dg^\beta)$$

with the gravitational energy-flow along the Killing vector field  $\xi \in \sec TM$ , then

$$\delta \mathcal{W}_\xi = \langle i_\xi dg_\alpha | \mathcal{T}^\alpha + \mathcal{J}^\alpha \rangle.$$

It must be observed that by employing the same argument used in the proof of Lemma 43, one can express the above force law entirely in terms of the components of the 2-form of non-metricity together with the matter currents ( $\mathcal{J}_\alpha$ ). In this way, the coupling of matter with non-metricity in our formalism can still be analyzed in more details.

Finally, as Nester, Adak and collaborators, we proved that a gravitational theory equivalent to GR can be formulated so that the gravitational field derives purely from the non-metricity of a flat torsionless connection. Particularly, we showed that our gravitational field equations coupled to the matter currents ( $\mathcal{J}_\alpha$ ) can be rewritten as

$$\begin{aligned} i_\nu \mathcal{J}_\mu = & \mathbf{Q}_{\mu[\alpha\nu];}^\alpha + \mathbf{Q}_{\alpha\nu}^\alpha{}_{;\mu} - \left[ \mathbf{Q}_{\alpha\beta}^{\alpha;\beta} + \frac{1}{2} \left( \mathbf{Q}_{\alpha\beta\gamma} \mathbf{Q}^{\alpha[\beta\gamma]} + \mathbf{Q}_{\alpha\beta}^\alpha \mathbf{Q}_\gamma^{\beta\gamma} \right) \right] \eta_{\mu\nu} \\ & - \mathbf{Q}_{\alpha[\mu\beta]} \mathbf{Q}^{\alpha[\beta\gamma]} \eta_{\gamma\nu} - \mathbf{Q}_{\mu\alpha\beta} \mathbf{Q}_\nu^{\alpha\beta} - \mathbf{Q}_{\mu[\alpha\nu]} \mathbf{Q}_\beta^{\alpha\beta}, \end{aligned}$$

where  $\mathbf{Q}_{\alpha\beta\gamma}$  are the components of the non-metricity 2-form of the NM connection  $\mathfrak{D}$  to which our gravitational potentials are adapted. From this, we see that the linearized field equations in terms of non-metricity are

$$i_\nu \mathcal{J}_\mu = \mathbf{Q}_{\mu[\alpha\nu];}^\alpha + \mathbf{Q}_{\alpha\nu}^\alpha{}_{;\mu} - \mathbf{Q}_{\alpha\beta}^{\alpha;\beta},$$

which constitute a coupled system of first order partial differential equations.

Now we close by commenting the following remark by Nester [25].

“Of course the STGR<sup>7</sup> formulation has some liabilities. It must be emphasized that in this geometry it is no longer possible to simply commute derivatives and the raising or lowering of indices via the metric as we are so accustomed to do in the standard Riemannian approach. Hence tensorial equations will appear differently depending on how the indices are arranged. (...) Another obvious limitation of the STGR formulation is that it (almost) requires a global coordinate system”.

---

<sup>7</sup>“Symmetric Teleparallel General Relativity”. See p. 28.

The above criticisms are irrelevant to our non-metricity formulation of GR, as we have adopted the calculus of differential forms instead of the classical tensorial calculus. We assumed that our spacetime manifold is parallelizable (something which can be justified physically from the existence of spinorial fields) and that the gravitational potentials are represented by a cobase field  $(g_\alpha)$ . The only components which we have utilized above are the components  $\mathbf{Q}_{\alpha\beta\gamma}$  relative to  $(g_\alpha)$  of the 2-form of non-metricity  $\mathbf{Q}_\gamma$  in Eq.(50), for which

$$\mathbf{Q}_\gamma = \frac{1}{2} \mathbf{Q}_{\alpha\beta\gamma} g^\alpha \wedge g^\beta.$$

As  $(g_\alpha)$  exists globally and our “raising and lowering” of indices derives from the metric  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$ , we can raise and lower indices in Eq.(50) with  $\mathbf{g}$  as usual, and no global coordinate system is required.

**Acknowledgment.** The author is grateful to Waldyr Rodrigues, for the important discussions during the development of this work, to Zbigniew Oziewicz, for having read and commented on the manuscript, and to Yen Chin and Muzaffer Adak, for having called my attention to the literature of the STGR.

## A Einstein-Hilbert Lagrangian

In this Appendix, we show how the usual interpretation of GR in terms of the curvature of the pseudo-Riemannian space  $(M, \mathbf{g}, \nabla)$ , where  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$  is induced by the gravitational potentials  $(g_\alpha)$ , arises from the Wallner Lagrangian. That is, we shall geometrize the theory in a Lorentzian space by giving a privilege to its Levi-Civita connection.

First, we show that the WL can be decomposed in three terms, one of which is of Yang-Mills type. Such decomposition is due to Rodrigues and de Souza [16].

**Lemma 48** *The WL can be written as*

$$\mathcal{L} = \frac{1}{2} dg_\alpha \wedge \star dg^\alpha - \frac{1}{2} \delta g_\alpha \wedge \star \delta g^\alpha - \frac{1}{4} g_\alpha \wedge dg^\alpha \wedge \star (g_\beta \wedge dg^\beta).$$

In fact, using Lemma 13,

$$\begin{aligned} \delta g_\alpha \wedge \star \delta g^\alpha &= i_\alpha dg^\alpha \wedge \star i_\beta dg^\beta = -dg^\alpha \wedge i_\alpha \star i_\beta dg^\beta \\ &= -dg^\alpha \wedge \star (i_\beta dg^\beta \wedge g_\alpha) \\ &= -dg^\alpha \wedge \star i_\beta (dg^\beta \wedge g_\alpha) + dg^\alpha \wedge \star dg_\alpha \\ &= -i_\beta (dg^\beta \wedge g_\alpha) \wedge \star dg^\alpha + dg^\alpha \wedge \star dg_\alpha \\ &= -g_\alpha \wedge dg^\beta \wedge \star (g_\beta \wedge dg^\alpha) + dg^\alpha \wedge \star dg_\alpha. \blacksquare \end{aligned}$$

Now we geometrize the gravitational theory.

**Lemma 49** *Let  $M$  be a four-dimensional parallelizable manifold and  $(g_\alpha)$  a tetrad on  $M$  such that  $\mathbf{g} = \eta_{\alpha\beta} g^\alpha \otimes g^\beta$  is a Lorentzian metric. Let  $(\mathcal{R}^\alpha_\beta)$  be the curvature 2-forms of the Levi-Civita connection of  $(M, \mathbf{g})$ ,  $(\mathcal{R}_\alpha)$  the Ricci 1-forms and  $\mathcal{R} = i_\alpha \mathcal{R}^\alpha$  the Ricci scalar. Therefore, the WL can be written (up to an exact differential) as*

$$\mathcal{L} = \frac{1}{2} \mathcal{R} \star 1 = -\frac{1}{2} \mathcal{R}_{\alpha\beta} \wedge \star (g^\alpha \wedge g^\beta).$$

Before we start our geometrization, let's prove the second equality of the latter equation. Indeed, by Eq.(1),  $\mathcal{R}_\alpha = -i_\beta \mathcal{R}^\beta_\alpha$ , so that

$$\mathcal{R} = -i_\alpha i_\beta \mathcal{R}^{\beta\alpha} = -i_{g_\beta \wedge g_\alpha} \mathcal{R}^{\beta\alpha} = -i_{g^\alpha \wedge g^\beta} \mathcal{R}^{\alpha\beta},$$

and therefore

$$\mathcal{R} \star 1 = -\star (i_{g^\alpha \wedge g^\beta} \mathcal{R}^{\alpha\beta}) = -\mathcal{R}_{\alpha\beta} \wedge \star (g^\alpha \wedge g^\beta).$$

We remark that the above minus sign derives from our definition of the Ricci tensor.

Now, let  $(\theta^\alpha_\beta)$  be the Levi-Civita connection 1-forms of  $(M, g)$  relative to  $(g_\alpha)$ . Recalling the first Cartan structural equation,  $dg_\alpha = -\theta^\beta_\alpha \wedge g_\beta$ , we can prove that

$$\begin{aligned} & 2dg_\alpha \wedge \star dg^\alpha - \frac{1}{2} g_\alpha \wedge dg^\alpha \wedge \star (g_\beta \wedge dg^\beta) \\ &= -(\theta_{\alpha\gamma} \wedge g^\gamma) \wedge \star dg^\alpha - (\theta_{\alpha\gamma} \wedge g^\gamma) \wedge \star dg^\alpha + \frac{1}{2} g^\alpha \wedge (\theta_{\alpha\gamma} \wedge g^\gamma) \wedge \star (g_\beta \wedge dg^\beta) \\ &= -\theta_{\alpha\gamma} \wedge \star dg^\alpha \wedge g^\gamma + \theta_{\alpha\gamma} \wedge \star dg^\gamma \wedge g^\alpha - \frac{1}{2} \theta_{\alpha\gamma} \wedge \star (g_\beta \wedge dg^\beta) \wedge g^\alpha \wedge g^\gamma \\ &= -\theta_{\alpha\gamma} \wedge \star^2 (\star dg^\alpha \wedge g^\gamma) + \theta_{\alpha\gamma} \wedge \star^2 (\star dg^\gamma \wedge g^\alpha) - \frac{1}{2} \theta_{\alpha\gamma} \wedge \star^2 (\star (g_\beta \wedge dg^\beta) \wedge g^\alpha \wedge g^\gamma) \\ &= -\theta_{\alpha\gamma} \wedge \star i^\gamma (\star^2 dg^\alpha) + \theta_{\alpha\gamma} \wedge \star i^\alpha (\star^2 dg^\gamma) - \frac{1}{2} \theta_{\alpha\gamma} \wedge \star i^\gamma i^\alpha \star^2 (g_\beta \wedge dg^\beta) \\ &= \theta_{\alpha\gamma} \wedge \star i^\gamma dg^\alpha - \theta_{\alpha\gamma} \wedge \star i^\alpha dg^\gamma - \frac{1}{2} \theta_{\alpha\gamma} \wedge \star i^\gamma i^\alpha (g_\beta \wedge dg^\beta) \\ &= \theta_{\alpha\gamma} \wedge \star \left[ i^\gamma dg^\alpha - i^\alpha dg^\gamma + \frac{1}{2} i^\alpha i^\gamma (g_\beta \wedge dg^\beta) \right]. \end{aligned}$$

On the other hand, by Eq.(7) (also, cf. Lemma 14),

$$\theta^{\alpha\gamma} = i^\gamma dg^\alpha - i^\alpha dg^\gamma + \frac{1}{2} i^\alpha i^\gamma (g_\beta \wedge dg^\beta),$$

so that

$$2dg_\alpha \wedge \star dg^\alpha - \frac{1}{2} g_\alpha \wedge dg^\alpha \wedge \star (g_\beta \wedge dg^\beta) = \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma}. \quad (51)$$

Also,

$$\begin{aligned}
-dg_\alpha \wedge \star dg^\alpha &= -(-\theta_{\alpha\gamma} \wedge g^\gamma) \wedge \star(-\theta_\delta^\alpha \wedge g^\delta) \\
&= -(\theta_{\gamma\alpha} \wedge g^\gamma) \wedge \star(-\theta_\delta^\alpha \wedge g^\delta) \\
&= \theta_{\alpha\gamma} \wedge g^\alpha \wedge \star(\theta_\delta^\gamma \wedge g^\delta),
\end{aligned} \tag{52}$$

and, using the proof of Lemma (12),

$$\begin{aligned}
-\delta g_\alpha \wedge \star \delta g^\alpha &= -\delta g_\alpha \wedge \star(-\star d \star g^\alpha) = \delta g_\alpha \wedge \star^2(d \star g^\alpha) \\
&= -\delta g_\alpha \wedge d \star g^\alpha = \star(d \star g_\alpha) \wedge d \star g^\alpha \\
&= d \star g^\alpha \wedge \star(d \star g_\alpha) \\
&= \theta_\gamma^\alpha \wedge \star g^\gamma \wedge \star(\theta_{\alpha\delta} \wedge \star g^\delta).
\end{aligned} \tag{53}$$

So, by Eqs.(51), (52) and (53), the EHL becomes

$$\begin{aligned}
2\mathcal{L} &= \left[ 2dg_\alpha \wedge \star dg^\alpha - \frac{1}{2}(g_\alpha \wedge dg^\alpha) \wedge \star(g_\beta \wedge dg^\beta) \right] \\
&\quad - dg_\alpha \wedge \star dg^\alpha - \delta g_\alpha \wedge \star \delta g^\alpha \\
&= \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} + \theta_{\alpha\gamma} \wedge g^\alpha \wedge \star(\theta_\delta^\gamma \wedge g^\delta) + \theta_\gamma^\alpha \wedge \star g^\gamma \wedge \star(\theta_{\alpha\delta} \wedge \star g^\delta).
\end{aligned} \tag{54}$$

Now, the third term can be written as

$$\begin{aligned}
&\theta_\gamma^\alpha \wedge \star g^\gamma \wedge \star(\theta_{\alpha\delta} \wedge \star g^\delta) \\
&= \star \theta_\gamma^\alpha \wedge g^\gamma \wedge \star(\star \theta_{\alpha\delta} \wedge g^\delta) \\
&= -\star^2(\star \theta_\gamma^\alpha \wedge g^\gamma) \wedge \star(\star \theta_{\alpha\delta} \wedge g^\delta) \\
&= -\star(i^\gamma \theta_\gamma^\alpha \wedge i^\delta \theta_{\alpha\delta}),
\end{aligned} \tag{55}$$

while the first two as

$$\begin{aligned}
&\theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} + \theta_{\alpha\gamma} \wedge g^\alpha \wedge \star(\theta_\delta^\gamma \wedge g^\delta) \\
&= \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} + \theta_{\alpha\gamma} \wedge g^\alpha \wedge i^\delta \star \theta_\delta^\gamma \\
&= \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} - i^\delta(\theta_{\alpha\gamma} \wedge g^\alpha) \wedge \star \theta_\delta^\gamma \\
&= \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} - \theta_{\alpha\gamma} \wedge \star \theta^{\alpha\gamma} - i^\delta \theta_{\alpha\gamma} \wedge g^\alpha \wedge \star \theta_\delta^\gamma \\
&= i^\delta \theta_{\alpha\gamma} \wedge \star \theta_\delta^\gamma \wedge g^\alpha \\
&= -i^\delta \theta_{\alpha\gamma} \wedge \star^2(\star \theta_\delta^\gamma \wedge g^\alpha) \\
&= -i^\delta \theta_{\alpha\gamma} \wedge \star i^\alpha \theta_\delta^\gamma \\
&= \star(i^\delta \theta_{\alpha\gamma} \wedge i^\gamma \theta_\delta^\alpha).
\end{aligned} \tag{56}$$

Therefore, by Eqs.(54), (55) and (56),

$$\begin{aligned}
2\mathcal{L} &= \star(i^\delta \theta_{\alpha\gamma} \wedge i^\gamma \theta_\delta^\alpha - i^\gamma \theta_\gamma^\alpha \wedge i^\delta \theta_{\alpha\delta}) \\
&= -i_{g^\gamma \wedge g^\delta}(\theta_{\alpha\gamma} \wedge \theta_\delta^\alpha) \star 1 \\
&= -(\theta_{\alpha\gamma} \wedge \theta_\delta^\alpha) \wedge \star(g^\gamma \wedge g^\delta)
\end{aligned}$$



or simply that

$$\mathcal{L} = \frac{1}{2} \theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}).$$

However,

$$\begin{aligned} d\theta_{\gamma\delta} \wedge \star (g^{\gamma} \wedge g^{\delta}) &= d(\theta_{\gamma\delta} \wedge \star (g^{\gamma} \wedge g^{\delta})) + \theta_{\gamma\delta} \wedge d\star (g^{\gamma} \wedge g^{\delta}) \\ &= d(\dots) + \theta_{\gamma\delta} \wedge [-\theta_{\epsilon}^{\gamma} \wedge \star (g^{\epsilon} \wedge g^{\delta}) - \theta_{\epsilon}^{\delta} \wedge \star (g^{\gamma} \wedge g^{\epsilon})] \\ &= d(\dots) - 2\theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}). \end{aligned}$$

Now, the reader must remember the second Cartan structural equation, so that the above equation yields

$$\begin{aligned} \mathcal{R}_{\gamma\delta} &= d\theta_{\gamma\delta} \wedge \star (g^{\gamma} \wedge g^{\delta}) + \theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}) \\ &= d(\dots) - 2\theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}) + \theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}) \\ &= d(\dots) - \theta_{\gamma\alpha} \wedge \theta_{\delta}^{\alpha} \wedge \star (g^{\gamma} \wedge g^{\delta}). \end{aligned}$$

Therefore, in terms of curvature,

$$\mathcal{L} = \frac{1}{2} d(\dots) - \frac{1}{2} \mathcal{R}_{\alpha\beta} \wedge \star (g^{\alpha} \wedge g^{\beta}). \blacksquare$$

## References

- [1] Hicks, N. J., *Notes on Differential Geometry*, Van Nostrand Reinhold Company, Amsterdam, 1965.
- [2] O'Neill, B., *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.
- [3] Thirring, W., *Classical Mathematical Physics*, Springer-Verlag, New York, 1997.
- [4] G  ckeler, M. & Sch  cker, T., *Differential Geometry, Gauge Theories and Gravity*, Cambridge University Press, Cambridge, 1989.
- [5] Rodrigues Jr., W. A., de Oliveira, E. C., *The Many Faces of Maxwell, Dirac and Einstein Equations*, Lectures Notes Physics **722**, Springer, Heidelberg, 2007. A preliminary and improved version may be found at <http://www.ime.unicamp.br/~walrod/recentes.htm>.
- [6] Rodrigues Jr, W. A., Fern  ndez, V. V., *Gravitation as a Plastic Distortion of the Lorentz Vacuum*, Fundamental Theories of Physics **168**, Springer, Heidelberg, 2010. A new version with corrections may be found at <http://www.ime.unicamp.br/~walrod/recentes.htm>.
- [7] Burke, W. L., *Applied Differential Geometry*, Cambridge University Press, Cambridge, 1985.

- [8] Anderson, J. L., *Principles of Relativity Physics*, Academic Press, New York, 1967.
- [9] Landau, L. D., Lifshitz, E. M., *The Classical Theory of Fields*, Butterworth-Heinemann, Oxford, 1973.
- [10] Aldrovandi, R., Pereira, J. G., *Teleparallel Gravity*, Fundamental Theories of Physics **173**, Springer, Heidelberg, 2013. An online version may be found at [www.ift.unesp.br/users/jpereira/tele.pdf](http://www.ift.unesp.br/users/jpereira/tele.pdf).
- [11] Wallner, R. P., Notes on Gauge Theory and Gravitation, *Acta Phys. Austriaca* **54**, 165–189 (1981).
- [12] Thirring, W., Wallner, R., The Use of Exterior Forms in Einstein’s Gravitation Theory, *Brazilian Journal of Physics* **8**, 686–723 (1978).
- [13] Thirring, W., Gauge Theories of Gravitation, *Acta Phys. Austriaca*, Suppl. XIX 439–462 (1978).
- [14] Blagojevic, M., Hehl, F. W., *Gauge Theories of Gravitation. A Reader with Commentaries*, Imperial College Press, London 2013. [[arXiv:1210.3775](https://arxiv.org/abs/1210.3775) [[gr-qc](#)]]
- [15] Sparling, G. A. J., Twistors, Spinors and the Einstein Vacuum Equations, *Preprint of the University of Pittsburgh* (1982).
- [16] Rodrigues, Jr., W. A., de Souza, Q. A. G., The Clifford Bundle and the Nature of the Gravitational Field, *Foundations of Physics* **23**, 1465–1490 (1993).
- [17] Rodrigues, Jr., W. A., The Nature of Gravitational Field and its Legitimate Energy-Momentum Tensor, *Reports on Mathematical Physics* **69**, 275–279 (2011). [[arXiv:1109.5272](https://arxiv.org/abs/1109.5272) [[math-ph](#)]]
- [18] Bramson, B., Do electromagnetic waves harbour gravitational waves?, *Proc. R. Soc. A* **462**, 1987–2000 (2006).
- [19] Vaidya, P. C., Unified Gravitational and Electromagnetic Waves, *Progress of Theoretical Physics* **25**, 305–314 (1961).
- [20] Roy, S. R., Tripathi, V. N., On Gravitational and Electromagnetic Waves in General Relativity, *General Relativity and Gravitation* **5**, 257–274 (1974).
- [21] Notte-Cuello, E. A., da Rocha, R., Rodrigues, Jr., W. A., Some Thoughts on Geometries and on the Nature of the Gravitational Field, *J. Phys. Math* **2**, 20–40 (2010). [[arXiv:0907.2424](https://arxiv.org/abs/0907.2424) [[math-ph](#)]]
- [22] Mol, I., Revisiting the Schwarzschild and the Hilbert-Droste Solutions of Einstein Equation and the Maximal Extension of the Latter (2014) [[arXiv:1403.2371](https://arxiv.org/abs/1403.2371) [[math-ph](#)]]

- [23] Baekler, P., Hehl, F. W., Miele, E. W., Nonmetricity and Torsion: Facts and Fancies in Gauge Approaches to Gravity, in Ruffini, R. (ed.), *Proc. 4th. Marcel Grossman Meeting on General Relativity* pp. 277–316, North-Holland, Amsterdam, 1986.
- [24] Maluf, J. W., Hamiltonian Formulation of the Teleparallel Description of General Relativity, *J. Math. Phys.* **35**, 335–343 (1994).
- [25] Nester, J. M., Yo, H.-J., Symmetric Teleparallel General Relativity, *Chinese J. Phys.* **37**, 113–118 (1999). [[arXiv:gr-qc/9809049](#)]
- [26] Adak, M., The Symmetric Teleparallel Gravity, *Turk. J. Phys* **30**, 379–390 (2006). [[arXiv:gr-qc/0611077](#)]
- [27] Adak, M., A Note on Parallel Transportation in Symmetric Teleparallel Geometry (2011). [[arXiv:1102.1878](#) [[physics.gen-ph](#)]]
- [28] Adak, M., Sert, O., Kalay, M., Sari, M., Symmetric Teleparallel Gravity: Some Exact Solutions and Spinor Couplings, *International Journal of Modern Physics A* **28** (2013). [[arXiv:0810.2388](#) [[gr-qc](#)]]
- [29] Adak M., Sert, O., A Solution to Symmetric Teleparallel Gravity, *Turk. J. Phys.* **29**, 1–7 (2005). [[arXiv:gr-qc/0412007](#)]